MA 430 HW2

8. Let $\vec{B} = \alpha x \hat{i}$ where $\alpha > 0$

$0 = \vec{\nabla} \cdot \vec{B} = \frac{\partial B_i}{\partial x} = \alpha \neq 0$

$\vec{B}$ doesn't obey Maxwell's equations, ∴ doesn't exist (as a magnetic field)

9. $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, $V' = V - \frac{\partial \lambda}{\partial t}$

$\vec{B} = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \nabla\lambda) = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \nabla\lambda}_{=0} = \vec{\nabla} \times \vec{A}$

Idenity: $\vec{\nabla} \times (\vec{\nabla} f) = 0$

$\vec{E} = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}(V - \frac{\partial \lambda}{\partial t}) - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\lambda)$

$= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} + \cancel{\frac{\partial}{\partial t}(\vec{\nabla}\lambda)} - \cancel{\frac{\partial}{\partial t}(\vec{\nabla}\lambda)} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ ✓

10. $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0$ $\quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t})$

$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \mu_0 \epsilon_0 \vec{\nabla}(\frac{\partial V}{\partial t})$

also $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{\nabla} \cdot (-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}) = \frac{\rho}{\epsilon_0}$

$-\vec{\nabla} \cdot \vec{\nabla} V = \frac{\rho}{\epsilon_0} + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A})$

$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A})$

11. a. $\vec{\nabla}(0) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \mu_0 \epsilon_0 \vec{\nabla}(\frac{\partial V}{\partial t})$

$\nabla^2 V = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}(0)$

$\Rightarrow \Box^2 \vec{A} = -\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} V)$, $\vec{\nabla}^2 V = \frac{-\rho}{\epsilon_0}$

b. $\nabla^2 V = \frac{-\rho}{\epsilon_0} - \frac{\partial}{\partial t}(-\mu_0 \epsilon_0 \frac{\partial V}{\partial t}) = \frac{-\rho}{\epsilon_0} + \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2}$

$\vec{\nabla}(-\mu_0 \epsilon_0 \frac{\partial V}{\partial t}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \mu_0 \epsilon_0 \vec{\nabla}(\frac{\partial V}{\partial t})$

$\cancel{-\mu_0 \epsilon_0 \vec{\nabla} \frac{\partial V}{\partial t}} - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \cancel{- \mu_0 \epsilon_0 \vec{\nabla}(\frac{\partial V}{\partial t})}$

$-\nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$

$\Rightarrow \Box^2 V = \frac{-\rho}{\epsilon_0}$

$\Box^2 \vec{A} = -\mu_0 \vec{J}$ $\quad \left(\text{where } \Box^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2\right)$