Ma 430 : Problems on Special Relativity

Problem 18 Given that $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and

$$x' = \gamma(x - \beta t)$$
$$x = \gamma(x' + \beta t')$$

Show that these equations can be solved to yield $\boxed{t' = \gamma(t - \beta x)}$

Problem 19 Show that if the matrices below are the matrices of linear transformations on $\mathbb{R}^4$ then they correspond to Lorentz transformations. In each of the matrices below $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and $R \in O(3)$ (so $R^T R = I$),

$$B_x = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B_z = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \qquad \tilde{R} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right].$$

- Use Prop. 7.6.2 which says if $L(x) = Ax$ then $L \in \mathcal{L} \Leftrightarrow A^T \eta A = \eta$.)
- Also to prove $\tilde{R}^T \eta R = \eta$ use "block multiplication", I remind or introduce to you the fact that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right] = \left[\begin{array}{c|c} AE + BG & AF + BH \\ \hline CE + DG & CF + DH \end{array}\right]$$

where A, B, C, D, E, F, G, H are matrices. For example ηR is calculated,

$$\left[\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & IR & \\ 0 & & & \end{array}\right].$$

Problem 20 Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^4$ be the trajectory of a light beam. Let us <u>define</u> that the velocity of γ with respect to the inertial frame $\mathcal{X}$ is $V_{\mathcal{X}}(t) \equiv \frac{d}{dt}\left(\vec{\mathcal{X}}(\gamma(t))\right)$ where $\mathcal{X} = (\mathcal{X}^0, \vec{\mathcal{X}})$ so that $V_{\mathcal{X}}$ is a 3-vector. Now show that $\| V_{\mathcal{X}}(t) \| = c \Rightarrow \| V_{\bar{\mathcal{X}}}(\bar{t}) \| = c$ for $\bar{\mathcal{X}} = B_x \mathcal{X}$ and $\bar{\mathcal{X}} = \tilde{R}\mathcal{X}$.

(BONUS)

$\boxed{\text{PROBLEM 21}}$ Let $\gamma: I \subset \mathbb{R} \to \mathbb{R}^3$ be the trajectory of a light beam. As usual the velocity of the light beam with respect to the inertial observer χ is $V_\chi = \frac{d}{dt}(\vec{\chi}(\gamma(t)))$ where I've seperated the three spatial directions since χ actually has 4 component maps and $\chi = (\chi^0, \vec{\chi})^T$.

If $\bar{\chi}$ is another inertial observer then by definition there exists a 4×4 matrix $\Lambda \in SO(1,3)$ $(\Lambda^T \eta \Lambda = \eta)$ such that $\bar{\chi} = \Lambda \chi$. Prove the following

$$\boxed{\| V_\chi \|^2 = c^2 \Rightarrow \| V_{\bar{\chi}} \|^2 = c^2} \quad (*)$$

where c is a constant and $V_{\bar{\chi}} = \frac{d}{d\bar{t}}(\vec{\bar{\chi}}(\gamma(t)))$. Arguing directly might be hard so you might try the following

1.) Show that $(*)$ holds for $\bar{\chi} = \tilde{R}\chi$ where $\tilde{R}$ is a rotation as in problem 19.

2.) Show that $(*)$ holds for $\bar{\chi} = B_x \chi$ where B_x is x-boost.

3.) Argue that any Lorentz transformation can be written as a composition of 1.) and 2.). I'm not sure how hard this is to show for arbitrary Λ.

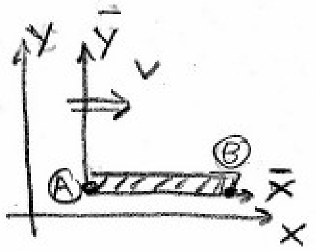
<u>**Remark**</u>: this problem is a <u>bonus</u> problem. It may be difficult to show what I've asked here. $\boxed{\text{PROBLEM 20}}$ is a proof of $\boxed{\text{PROBLEM 21}}$ by example but we'd like to do it for arbitrary $\Lambda \in SO(1,3)$. This would show Lorentz transformations preserve the constant speed of light. There is a physicsy proof, $\frac{dx}{dt} = c$ then $dx = c\,dt$ and the infinitesimal interval is $-c^2dt^2 + dx^2 = ds^2 = -c^2d\bar{t}^2 + d\bar{x}^2 = d\bar{s}^2$ which is zero for light since $dx^2 = c^2dt^2$ thus

$$-c^2d\bar{t}^2 + d\bar{x}^2 = 0 \Rightarrow \left(\frac{d\bar{x}}{d\bar{t}}\right)^2 = c^2 \therefore \frac{d\bar{x}}{d\bar{t}} = c.$$

I'd like to find a better more mathematical proof.

$\boxed{\text{PROBLEM 22}}$ Consider a rod which is measured to have length L_0 when at rest. Put the rod in motion such that it resides at the origin of the $\bar{x}$ where

$$\bar{x} = \begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \Lambda x$$



$$x = \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix} = \Lambda^{-1} \bar{x}$$

this makes $\bar{x}$ the x-boosted frame w.r.t. x. Now measure the length of the rod w.r.t. x. We assume

$$\bar{x}(A) = (\bar{t}_A, 0) = (\bar{t}_A, \bar{x}_A)$$
$$\bar{x}(B) = (\bar{t}_B, L_0) = (\bar{t}_B, \bar{x}_B)$$

When measuring length we find $x'(A)$ and $x'(B)$ at the same time t, this means that $\bar{t}_A \neq \bar{t}_B$ but rather

$$t = \gamma(\bar{t}_A + \beta \bar{x}_A) = \gamma(\bar{t}_B + \beta \bar{x}_B)$$
$$\underline{\bar{t}_A = \bar{t}_B + \beta L_0}$$

Note $x'(A) = \gamma(\bar{x}_A + \beta \bar{t}_A)$ and $x'(B) = \gamma(\bar{x}_B + \beta \bar{t}_B)$.

$\boxed{\text{Show } x'(B) - x'(A) = L_0/\gamma}$ this shows that a rod of length one-meter will be observed to only be $1/2$ meter long if it was moving so fast that $\gamma = 2$.

$\boxed{\text{PROBLEM 23}}$ Consider a fixed point in the moving $\bar{x}$-frame as in $\boxed{\text{PROBLEM 22}}$. Suppose a clock at $\bar{x} = \bar{x}_0$ ticks from $\bar{t}_1$ to time $\bar{t}_2$. We label these two "events" by

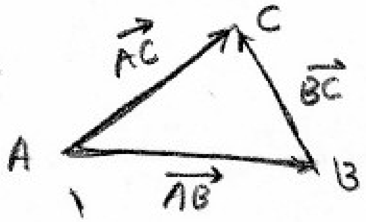
$$\bar{x}(A) = (\bar{t}_1, \bar{x}_0) \qquad \bar{x}(B) = (\bar{t}_2, \bar{x}_0)$$

$\boxed{\text{Show that } x^0(B) - x^0(A) = \gamma(\bar{t}_2 - \bar{t}_1)}$. In other words $\Delta t = \gamma \Delta \bar{t}$, this shows that a radioactive particle with a 1-second liftime will be observed to have a 10-second liftime if it has $\gamma = 10$.

PROBLEM 24 Find a <u>counter-example</u> to the triangle-inequality for the Minkowski metric. In particular find $A, B, C \in \mathbb{R}^4$ such that, $\overline{AB} = (B^0 - A^0, B^1 - A^1, B^2 - A^2, B^3 - A^3)$ and,

$$\langle \overline{AB}, \overline{AB} \rangle + \langle \overline{BC}, \overline{BC} \rangle \leq \langle \overline{AC}, \overline{AC} \rangle$$

(ok, btw we really should define $\|x\| = \sqrt{|\langle x, x\rangle|}$ and show $\|\overline{AB}\| + \|\overline{BC}\| \leq \|\overline{AC}\|$ but we'll settle for what I've assigned.) Ordinarily for genuine math norms $\|\cdot\|$,



$$\|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| \geq \|\overrightarrow{AC}\|$$

PROBLEM 25 If we have two events $A = (t_A, x_A, y_A, z_A)$ and $B = (t_B, x_B, y_B, z_B)$ which are seperated by a time like vector $(\langle B-A, B-A\rangle < 0)$ then $\boxed{\text{show } t_A < t_B \Rightarrow \bar{t}_A < \bar{t}_B}$ where $\bar{t}$ is the time in the x-boosted frame relative to the frame where we have described A and B,

$$\begin{aligned} \bar{t} &= \gamma(t - \beta x) \\ \bar{x} &= \gamma(x - \beta t) \\ \bar{y} &= y \\ \bar{z} &= z \end{aligned}$$

PROBLEM 26 Prove Proposition 7.10.6. (a few lines)

PROBLEM 27 Prove Proposition 7.10.7. (follow my derivation of the non-relativistic case, the math is the same modulo an extra γ-factor and the starting point appeals to Prop. 7.10.6)