Due Thursday, Oct 19.

$\boxed{\text{PROBLEM 28}}$ Let V be a vector space with $\{e_i\}_{i=1}^n$ an ordered basis. Define the dual-space to V

$$V^* = \{\alpha : V \to \mathbb{R} \mid \alpha \text{ a linear mapping}\}$$

Define dual basis $\{e^1, e^2, \dots, e^n\} = \{e^j\}_{j=1}^n$ by $e^j(e_i) = \delta_i^j$ where we assume $e^j : V \to \mathbb{R}$ are linear maps. We give V^* the structure of a vector space as follows,

$$(\alpha + \beta)(x) \equiv \alpha(x) + \beta(x)$$
$$(c\alpha)(x) \equiv c\,\alpha(x)$$

$\forall \alpha, \beta \in V^*$ and $\forall c \in \mathbb{R}$ and $x \in V$. Show,

(i.) V^* is closed under vector addition & scalar multiplication. The proof of the remaining axioms for vector space follow similarly, thus V^* is a vector space.

(ii.) $\{e^j\}_{j=1}^n$ forms a basis for V^*.

(iii) if $V = \mathbb{R}^n$ and $\{e_i\}$ are column vectors then for each $\alpha \in V^*$ $\exists\, a \in \mathbb{R}^n$ such that

$$\alpha(x) = a^T x \qquad \forall x \in \mathbb{R}^n.$$

$\boxed{\text{PROBLEM 29}}$ Let $\mathcal{B} = \{b : V \times V \to \mathbb{R} \mid b \text{ is bilinear}\}$

(a.) Show that $\mathcal{B}$ forms a vector space w.r.t. operations,

$$(b+m)(x,y) = b(x,y) + m(x,y)$$
$$(cb)(x,y) = c\,b(x,y)$$

$\forall m, b \in \mathcal{B}$ and $\forall x, y \in V$ and $c \in \mathbb{R}$. You may limit your proof to closure of vector addition and scalar multiplication.

(b.) Show that $\{e^i \otimes e^j\}_{i,j=1}^n$ forms a basis for $\mathcal{B}$.

(i.) Linear Independence: $c_{ij}\, e^i \otimes e^j = 0 \Rightarrow c_{ij} = 0 \ \forall i,j.$

(technically we should have (iii) to complete (ii) anyway)

(ii) SPANNING: $b \in \mathcal{B}$ then $\exists\, b_{ij}$ s.t. $b = b_{ij}\, e^i \otimes e^j$.

(iii) Show that $e^i \otimes e^j \in \mathcal{B}$ for all $i, j \in \{1, 2, \dots, n\}$.

$\boxed{\text{PROBLEM 30}}$ Verify the claim of Example. 9.1.8. That is show that $h : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$h(u, v, w) \equiv u \cdot (v \times w)$$

cross-product on $\mathbb{R}^3$

is an antisymmetric trilinear mapping. This means you must show linearity in each slot as well as six symmetry properties of h.

$\boxed{\text{PROBLEM 31}}$ Prove prop. 9.3.4. That is show $\{ e_i \otimes e_j \}_{i,j=1}^{n}$ forms a basis for the space of all bilinear forms on V^*. This means you need to prove for $\mathcal{B}^*$ all the same items as we did for $\mathcal{B}$ back in $\boxed{\text{PROBLEM 29 b}}$.