MA 430 – HOMEWORK ON TENSORS

PROBLEM 32 verify the claims of Example 9.5.8. That is show

$$T_i{}^{jk} \equiv g_{il} T^{ljk} = (T')_i{}^{jk}$$

$$T_{ij}{}^{k} \equiv g_{il} g_{jm} T^{lmk} = (T'')_{ij}{}^{k}$$

$$T_{ijk} \equiv g_{il} g_{jm} g_{kn} T^{lmn} = (T''')_{ijk}$$

where T', T'' and T''' are defined in the notes.

PROBLEM 33 Suppose we have a tensor $G = G_{\mu\nu} e^\mu \otimes e^\nu$ in Minkowski Space where

$$(G_{\mu\nu}) = \begin{bmatrix} a & b & c & d \\ 0 & 1 & 2 & 3 \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \end{bmatrix}$$

where $a, b, c, d, x, y, z \in \mathbb{R}$. Find $(G^{\mu\nu})$ and present your answer as a matrix.

PROBLEM 34 Let $F = F_{\mu\nu} e^\mu \otimes e^\nu$ be the field tensor defined in eq$^{\underline{n}}$ 9.15. with respect to the coordinate system with basis $\{e_\mu\}$ and dual basis $\{e^\nu\}$. Suppose $\{\bar{e}_\mu\}$ is another coordinate system related to the original coordinate system as

$$\bar{e}_\mu = (\Lambda^{-1})_\mu^{\nu} e_\nu \quad \text{and} \quad \bar{e}^\mu = \Lambda^\mu_{\ \nu} e^\nu$$

denote the components of F w.r.t new coordinates by $\bar{F}_{\mu\nu} \equiv F(\bar{e}_\mu, \bar{e}_\nu)$. Show that

$$\boxed{F_{\mu\nu} F^{\mu\nu} = \bar{F}_{\mu\nu} \bar{F}^{\mu\nu}}$$

(Your proof ought not involve explicit E_i or B_i we can do this at the level of $F_{\mu\nu}$)

PROBLEM 35 We now investigate what is implicitly claimed in writing the $\vec{E}$ and $\vec{B}$ fields in terms of the field tensor. Since we know how $F_{\mu\nu}$ transforms to $\bar{F}_{\mu\nu}$ this means we can see how E_i transforms to $\bar{E}_i$ and B_i transforms to $\bar{B}_i$ for $i=1,2,3$. The general transformation for an arbitrary Lorentz transformation is straightforward but ugly to write out in detail, instead let us consider an x-boost,

$$(\Lambda^\mu{}_\nu) = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the transformed $\boxed{\bar{F}_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F_{\alpha\beta}}$ —(★). We should expect

$$(\bar{F}_{\mu\nu}) = \begin{bmatrix} 0 & -\bar{E}_1 & -\bar{E}_2 & -\bar{E}_3 \\ \bar{E}_1 & 0 & \bar{B}_3 & -\bar{B}_2 \\ \bar{E}_2 & -\bar{B}_3 & 0 & \bar{B}_1 \\ \bar{E}_3 & \bar{B}_2 & -\bar{B}_1 & 0 \end{bmatrix}$$

if you study eq^{n} (★) you should be able to figure out how the electric and magnetic fields in the x-boosted frame relate to those in the original frame. You will find that the electric and magnetic fields in the barred frame are formed from a mixture of the $\vec{E}$ and $\vec{B}$.

$\boxed{\text{Find } \bar{E}_i \text{ and } \bar{B}_j \text{ for } i,j=1,2,3 \text{ in terms of } E_i,\ B_j \text{ and } \gamma, \beta.}$

PROBLEM 36 follow the method of Example 10.1.6 to find the formula for the determinant of the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix}$$