§10.3 # 1

$$\frac{dy}{dx} = \frac{y}{x} \overset{①}{\Rightarrow} \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \boxed{\ln|y| = \ln|x| + C}.$$

(implicit sol$^{\underline{n}}$)

We can solve for y here. Take exp of both sides,

$$e^{\ln|y|} = e^{\ln|x| + C} = e^{\ln|x|} e^C \Rightarrow |y| = e^C|x|$$

$$\Rightarrow y = \pm e^C x$$

$$\Rightarrow \boxed{y = kx \quad \text{for } k \neq 0}.$$

Notice $\boxed{y = 0}$ is also a sol$^{\underline{n}}$ here, we lost it in step ①.

§10.3 # 2

$$\frac{dy}{dx} = \frac{\sqrt{x}}{e^y} \Rightarrow \int e^y dy = \int \sqrt{x}\, dx$$

$$\Rightarrow e^y = \frac{2}{3}x^{3/2} + C$$

$$\Rightarrow \boxed{y = \ln\left(\frac{2}{3}x^{3/2} + C\right)}$$

§10.3 # 8

$$\frac{dy}{d\theta} = \frac{e^y \sin^2\theta}{y \sec\theta} \Rightarrow \underbrace{\int y e^{-y} dy}_{\text{I}} = \underbrace{\int \frac{\sin^2\theta}{\sec\theta} d\theta}_{\text{II}}$$

I: $\displaystyle\int \underbrace{y}_{u} \underbrace{e^{-y} dy}_{dv} = -ye^{-y} + \int e^{-y} dy = \underline{-ye^{-y} - e^{-y} + C_1}.$

II: $\displaystyle\int \frac{\sin^2\theta}{\sec\theta} d\theta = \int (\sin^2\theta)\cos\theta\, d\theta = \int u^2 du = \underline{\frac{1}{3}\sin^3\theta + C_2}.$

Thus, equating I & II and including an arbitrary constant,

$$\boxed{-ye^{-y} - e^{-y} = \frac{1}{3}\sin^3\theta + C}.$$

(I can't find a nice explicit sol$^{\underline{n}}$ here.)

$\boxed{\S 10.3 \# 11}$ Solve $\frac{dy}{dx} = \frac{x}{y}$ given $y(0) = -3$.

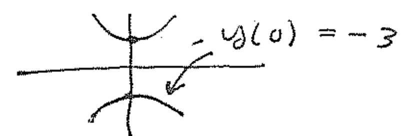
$$\frac{dy}{dx} = \frac{x}{y} \implies \int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y(0) = -3 \implies \frac{1}{2}(9) = \frac{1}{2}(0) + C \quad \therefore \quad C = \frac{9}{2}$$

$$\implies \frac{1}{2}y^2 = \frac{1}{2}y \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{9}{2}$$

$$\text{or} \quad \underbrace{y^2 - x^2 = 9}_{\text{hyperbola}}.$$



<u>We want the lower branch</u> $\implies \boxed{y = -\sqrt{9 + x^2}}$

$\boxed{\S 10.3 \# 12}$ Solve $y' = (y \cos x)/(1+y^2)$ given $y(0) = 1$

$$\frac{dy}{dx} = \frac{y\cos(x)}{1+y^2}$$

$$\implies \underbrace{\int \left(\frac{1+y^2}{y}\right) dy}_{\text{I}} = \underbrace{\int \cos(x)\,dx}_{\text{II.}}$$

Clearly (II.) $= \sin(x) + C_1$. Consider,

$$(\text{I.}) \quad \int \left(\frac{1+y^2}{y}\right) dy = \int \left(\frac{1}{y} + y\right) dy = \ln|y| + \frac{1}{2}y^2 + C_2$$

Thus,

$$\underline{\ln|y| + \frac{1}{2}y^2 = \sin(x) + C} \quad (\text{implicit general sol}^{n})$$

Apply the initial condition,

$$y(0) = 1 : \quad \ln(1) + \frac{1}{2} = \sin(0) + C \quad \therefore \quad \underline{C = 1/2}.$$

$$\therefore \quad \boxed{\ln|y| + \frac{1}{2}y^2 = \sin(x) + \frac{1}{2}}$$

$\boxed{\S 10.3 \# 16}$ Solve $xy' + y = y^2$ given $y(1) = -1$

$$x\frac{dy}{dx} = y^2 - y$$

$$\underbrace{\int \frac{dy}{y^2-y}}_{\text{I.}} = \underbrace{\int \frac{dx}{x}}_{\text{II}}$$

II $= \ln|x| + C_2$.

Notice $\frac{1}{y^2-y} = \frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$. We find A, B as usual: $1 = A(y-1) + By \Rightarrow \underline{1 = B}$, or $1 = -A \therefore \underline{A = -1}$.

Hence,

I: $\int \frac{dy}{y^2-y} = \int \left(\frac{-1}{y} + \frac{1}{y-1}\right) dy = \underline{-\ln|y| + \ln|y-1| + C}$.

Equating I & II yields,

$$-\ln|y| + \ln|y-1| = \ln|x| + C$$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = \ln|x| + C$$

$$\Rightarrow \ln\left|1 + \frac{1}{y}\right| = \ln|x| + C$$

$$\Rightarrow |1 - 1/y| = e^C |x| \Rightarrow \underbrace{1 - \frac{1}{y} = \pm e^C x}$$

$$\Rightarrow \frac{1}{y} = 1 + kx$$

$$\Rightarrow \underline{y = \frac{1}{1+kx}}.$$

Using the given $y(1) = -1 \Rightarrow -1 = \frac{1}{1+k} \Rightarrow \underline{k = -2}$.

$$\therefore \boxed{y = \frac{1}{1-2x}}$$

§10.3 #22 Solve $xy' = y + xe^{y/x}$ by subst. $v = y/x$

If $v = y/x$ then $y = vx$ thus $y' = v'x + v$. Hence,

$$\underbrace{x(v'x+v) = vx + xe^{v}}$$

← substituted v for y wherever possible using $y = vx$ and $y' = v'x + v$.

$$x^2v' + \cancel{vx} = \cancel{vx} + xe^{v}$$

$$x^2\frac{dv}{dx} = xe^{v}$$

$$\Rightarrow \int e^{-v}\,dv = \int \frac{1}{x}\,dx$$

$$-e^{-v} = \ln|x| + C \Rightarrow -e^{-\frac{y}{x}} = \ln|x| + C$$

$$\Rightarrow \exp\left(\frac{-y}{x}\right) = k - \ln|x|$$

$$\Rightarrow \frac{-y}{x} = \ln(k - \ln|x|)$$

$$\Rightarrow \boxed{y = -x\ln(k - \ln|x|)}.$$

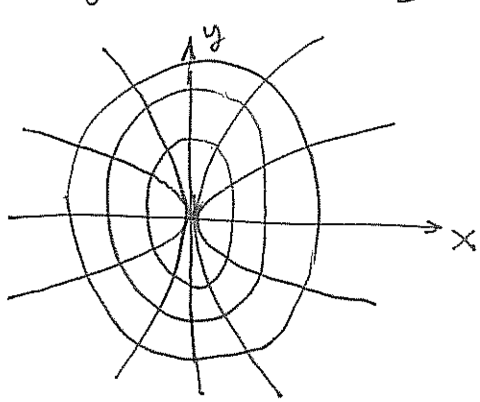
§10.3 #30 $y^2 = kx^3$ gives a family of curves. Find the O.T. for this family and sketch a few

If $y^2 = kx^3$ then $2\ln(y) = \ln(k) + 3\ln(x)$
hence these curves satisfy $\frac{2}{y}\frac{dy}{dx} = \frac{3}{x} \Rightarrow \frac{dy}{dx}\Big|_{curve} = \frac{3y}{2x}$

Orthogonal Traj will have $\frac{dy}{dx}\Big|_{O.T.} = \frac{-2x}{3y}$ hence solve

$$\int 3y\,dy = \int -2x\,dx \Rightarrow \frac{3}{2}y^2 = -x^2 + C \Rightarrow \boxed{\frac{y^2}{2/3} + x^2 = k}$$

the O.T. are ellipses!



(hopefully you found nicer pictures via Mathematica or some such technology.)