§ 10.5 #8 Solve $x^2 y' + 2xy = \cos^2 x$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{1}{x^2}\cos(x)$$: put into standard form

Calculate int. factor, $\mu = \exp\left(\int \frac{2}{x}\,dx\right) = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$.

Multiply by $\mu = x^2$ on the standard form DE^n,

$$\underbrace{x^2\frac{dy}{dx} + 2xy}_{} = \cos(x)$$

(integrate both sides w.r.t. x)

$$\frac{d}{dx}\left(x^2 y\right) = \cos(x) \implies x^2 y = \sin(x) + C$$

$$\therefore \boxed{y = \frac{1}{x^2}\sin(x) + \frac{C}{x^2}.}$$

§10.5 #10 Solve $y' + y = \sin(e^x)$

observe $P = 1$ and the given DE^n is already in standard form,

$$\mu = \exp\left(\int 1\,dx\right) = \exp(x) = e^x$$ ⟸ the integrating factor.

Multiply DE^n $\mu = e^x$,

$$\underbrace{e^x\frac{dy}{dx} + e^x y}_{} = e^x \sin(e^x)$$

$$\frac{d}{dx}\left(e^x y\right) = e^x\sin(e^x) \implies e^x y = \int e^x \sin(e^x)\,dx$$

$$\implies e^x y = \int \sin(u)\,du, \quad u = e^x$$

$$\implies e^x y = -\cos(e^x) + C$$

$$\therefore \boxed{y = -e^{-x}\cos(e^x) + Ce^{-x}}$$

§10.5 #17 Solve $\frac{dV}{dt} - 2tV = 3t^2 e^{t^2}$, $V(0) = 5$

Calculate $\mu = \exp\left(\int -2t\,dt\right) = \exp(-t^2)$. Multiply by the DE$_q^n$ by $\mu = e^{-t^2}$,

$$e^{-t^2}\frac{dV}{dt} - 2te^{-t^2}V = 3t^2 e^{t^2} e^{-t^2} = 3t^2$$

$$\frac{d}{dt}\left[e^{-t^2}V\right] = 3t^2$$

Integrate w.r.t. t both sides,

$$e^{-t^2}V = t^3 + C$$

$$V(t) = t^3 e^{t^2} + Ce^{t^2}.$$

$V(0) = 5 = 0 + C \therefore C = 5 \Rightarrow$ $\boxed{V(t) = t^3 e^{t^2} + 5e^{t^2}.}$

§10.5 #18 Solve $2xy' + y = 6x$, $x > 0$ and $y(4) = 20$

$$\frac{dy}{dx} + \left(\frac{1}{2x}\right)y = 3 \Rightarrow \mu = \exp\left(\int \frac{dx}{2x}\right) = e^{\frac{1}{2}\ln|x|} = e^{\ln|x|^{1/2}} = \sqrt{x}$$

assuming $x > 0$ as was given.

Multiply the DE$_q^n$ by $\mu = \sqrt{x}$,

$$\sqrt{x}\frac{dy}{dx} + \frac{1}{2\sqrt{x}}y = 3\sqrt{x}$$

$$\frac{d}{dx}\left[\sqrt{x}\,y\right] = \sqrt{x}$$

$$\sqrt{x}\,y = 3\left(\frac{2}{3}x^{3/2}\right) + C$$

$$y(x) = 2x + C/\sqrt{x}.$$

$y(4) = 2(4) + C/\sqrt{4} = 20 \Rightarrow \frac{C}{2} = 12 \Rightarrow C = 24.$

$$\therefore \boxed{y(x) = 2x + \frac{24}{\sqrt{x}}}$$

§10.5 #23 The Bernoulli DEq$^{\underline{n}}$ $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ is linear if $n=0$ or $n=1$. However, if $n>1$ then we can transform it to a linear DEq$^{\underline{n}}$ of the form

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)Q(x)$$

given the substitution $u = y^{1-n}$

Let $u = y^{1-n}$ then $y = u^{\frac{1}{1-n}}$ if $n \neq 1$.

Differentiate w.r.t. x;

$$\frac{dy}{dx} = \frac{1}{1-n}\,u^{\left(\frac{1}{1-n}-1\right)}\frac{du}{dx} = \left(\frac{1}{1-n}\right)u^{\frac{1-(1-n)}{1-n}}\frac{du}{dx}$$

Hence,

$$\frac{dy}{dx} = \left(\frac{1}{1-n}\right)u^{\frac{n}{1-n}}\frac{du}{dx}$$

Substitute into the Bernoulli DEq$^{\underline{n}}$,

$$\left(\frac{1}{1-n}\right)u^{\frac{n}{1-n}}\frac{du}{dx} + P(x)\,u^{\frac{1}{1-n}} = Q(x)\left(u^{\frac{1}{1-n}}\right)^n$$

$$\frac{du}{dx} + (1-n)\,u^{\left(\frac{1}{1-n}-\frac{n}{1-n}\right)}P(x) = (1-n)\,Q(x)$$

Note that $\frac{1}{1-n} - \frac{n}{1-n} = \frac{1-n}{1-n} = 1$. Hence, $\underline{\frac{du}{dx} + (1-n)\,u\,P = (1-n)\,Q}$ //

§10.5 #26 Substitute $u = y'$ to solve the following DEq$^{\underline{n}}$:

$$xy'' + 2y' = 12x^2 \Rightarrow xu' + 2u = 12x^2$$

$$\Rightarrow \frac{du}{dx} + \frac{2}{x}u = 12x$$

$$\Rightarrow \mu = \exp\left(\int \frac{2dx}{x}\right) = \exp\left(2\ln|x|\right) = \exp\left(\ln|x|^2\right) = x^2.$$

$$x^2\frac{du}{dx} + 2xu = 12x^3$$

Thus, $\frac{d}{dx}\left(x^2u\right) = 12x^3 \Rightarrow x^2u = 3x^4 + C_1$

$$\Rightarrow u = \frac{3x^4 + C_1}{x^2} = \frac{dy}{dx}$$ now integrate <u>again</u>

↲

§10.5 #26 Continued

$$\frac{dy}{dx} = 3x^2 + \frac{C_1}{x^2}$$

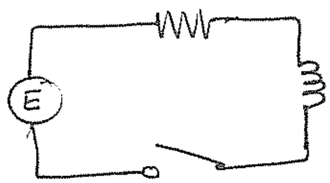
Integrate both sides w.r.t. x,

$$\boxed{y(x) = x^3 - \frac{C_1}{x} + C_2}$$

Remark: we have two arbitrary constants since this was a 2^{nd} order DEq$^{\underline{n}}$.

§10.5 # 27

FIG. 4



$$L\frac{dI}{dt} + RI = E(t)$$

Solve given that
$E(t) = 40V$ (constant)
$L = 2H$ and $R = 10\Omega$
and $I(0) = 0A$.

(a.) $$2\frac{dI}{dt} + 10I = 40$$

$$\frac{dI}{dt} + 5I = 20 \quad , \quad \mu = \exp\left(\int 5\,dt\right) = e^{5t}$$

$$e^{5t}\frac{dI}{dt} + 5e^{5t}I = 20e^{5t} \qquad \text{(multiplied by } \mu)$$

$$\frac{d}{dt}\left[e^{5t}I\right] = 20e^{5t}$$

$$e^{5t}I = \frac{20}{5}e^{5t} + C_1$$

$$I(t) = 4 + C_1e^{-5t}.$$

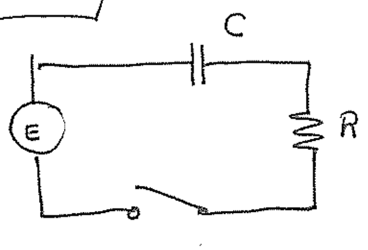
$$I(0) = 4 + C_1 = 0 \quad \therefore \quad C_1 = -4.$$

$$\therefore \quad \boxed{I(t) = 4 - 4e^{-5t}.}$$

[both in units of Amps.]

(b.) $$I(0.1) = 4 - 4e^{-5(0.1)} \cong \boxed{1.57 \cong I(0.1)}$$

§10.5 #29



$$RI + \frac{Q}{C} = E(t)$$

However, $I = dQ/dt$ thus

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Suppose $R = 5\,\Omega$, $C = 0.05\,F$, $E(t) = 60V$ and $Q(0) = 0$.
Find $Q(t)$ and $I(t)$

$$5\frac{dQ}{dt} + \frac{1}{0.05}Q = 60$$

$$\frac{dQ}{dt} + \frac{1}{0.25}Q = 12$$

$$\frac{dQ}{dt} + 4Q = 12 \Rightarrow \mu = e^{\int 4dt} = e^{4t}.$$

$$e^{4t}\frac{dQ}{dt} + 4e^{4t}Q = 12e^{4t}$$

$$\frac{d}{dt}\left[e^{4t}Q\right] = 12e^{4t}$$

$$e^{4t}Q = 3e^{4t} + C_1$$

$$\boxed{Q(t) = 3 + C_1e^{-4t}} \Rightarrow Q(0) = 0 = 3 + C_1$$

$$\therefore \underline{C_1 = -3}.$$

$$Q(t) = 3 - 3e^{-4t}$$

$$I(t) = \frac{dQ}{dt} = \frac{d}{dt}\left(3 - 3e^{-4t}\right)$$

$$\therefore \boxed{I(t) = 12e^{-4t}}$$

Remark: as $t \to \infty$ notice that $Q \to 3$ and $I \to 0$. For large times all the voltage is dropped across the capacitor which is like an open circuit once charged.