§12.1 #5 List first five terms in sequence $a_n = \dfrac{3(-1)^n}{n!}$

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, a_4, a_5, \dots\}$$
$$= \left\{\frac{-3}{1}, \frac{3}{2}, \frac{-3}{6}, \frac{3}{24}, \frac{-3}{120}, \dots\right\}$$

(I assume we begin at $n=1$ although it is ambiguous given Stewart's Problem statement)

§12.1 #15 List first 6 terms in $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{n}{2n+1}$. What limit does a_n tend towards? Find the limit.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \dots\right\} \longrightarrow \text{going towards } 1/2.$$

Observe that:

$$\lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{2+1/n}\right) = \boxed{\frac{1}{2}.}$$

§12.1 #22 Find limit of $a_n = 3^{n+2}/5^n$ as $n \longrightarrow \infty$.

$$\lim_{n\to\infty}\left(\frac{3^{n+2}}{5^n}\right) = \lim_{n\to\infty}\left(\frac{3^n \cdot 3^2}{5^n}\right)$$
$$= 9\lim_{n\to\infty}\left(\left(\frac{3}{5}\right)^n\right) \to 0$$
$$= \boxed{0.}$$

§12.1 #25 $a_n = \dfrac{(-1)^{n-1}\, n}{n^2+1}$ find $\lim\limits_{n\to\infty} a_n$, does it converge or diverge?

Notice $|a_n| = \dfrac{n}{n^2+1}$ and $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty}\left(\dfrac{n}{n^2+1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{1}{n+1/n}\right)$

thus $\lim\limits_{n\to\infty}|a_n| = 0$. By the absolute conv. Th$^{\underline{m}}$ for sequences we conclude $\boxed{\lim\limits_{n\to\infty}\left(\dfrac{(-1)^{n-1}\, n}{n^2+1}\right) = 0 \quad \text{(it converges to zero)}}$

$\boxed{\text{§12.1 \# 30}}$ Determine if $\{\arctan(2n)\}_{n=1}^{\infty}$ is convergent or divergent. If it's a convergent sequence then find its limit.

Observe that $f(x) = \tan^{-1}(2x)$ is a function of $\mathbb{R}$ which corresponds to $a_n = \tan^{-1}(2n)$ since $f(n) = a_n$ for all $n \in \mathbb{N}$. Observe that the inverse function for $f(x)$ (relative to restriction of tangent near zero) is found by the usual algorithm,

$$y = \tan^{-1}(2x) \longrightarrow x = \tan^{-1}(2y)$$

$$\longrightarrow \tan(x) = \tan(\tan^{-1}(2y))$$

$$\longrightarrow y = \tfrac{1}{2}\tan(x)$$

$$\longrightarrow \underline{f^{-1}(x) = \tfrac{1}{2}\tan(x)}.$$

Thus, the horizontal asymptotes for $\tan^{-1}(2x)$ are the same as those for $\tan^{-1}(x)$. They are $y = \pm\frac{\pi}{2}$. In particular $\tan^{-1}(2n) \longrightarrow \frac{\pi}{2}$ as $x \to \infty$. Hence, by correspondence Th$^{\underline{m}}$, $\boxed{\lim_{n\to\infty} \tan^{-1}(2n) = \frac{\pi}{2}}$ (it converges!)

Alternatively,

$$\lim_{n\to\infty}\left(\tan^{-1}(2n)\right) = \underbrace{\lim_{m\to\infty}\left(\tan^{-1}(m)\right)}_{\text{this is a "known" limit.}} = \frac{\pi}{2}$$

making the substitution $m = 2n$ where clearly $m \to \infty$ if $n \to \infty$.

$\boxed{\S 12.1 \# 33}$ Find $\lim_{n\to\infty}(n^2 e^{-n})$, does it converge or diverge?

Extend n to be a continuous variable in what follows,

$$\lim_{n\to\infty}(n^2 e^{-n}) = \lim_{n\to\infty}\left(\frac{n^2}{e^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{n\to\infty}\left(\frac{2n}{e^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{n\to\infty}\left(\frac{2}{e^n}\right) = \underline{0}.$$

Thus $a_n = n^2 e^{-n}$ converges to 0 as $n \to \infty$.

$\boxed{\S 12.1 \# 62}$ Let $a_n = \frac{2n-3}{3n+4}$. Is this sequence increasing, decreasing, monotonic? Is the sequence bounded?

There are many ways to argue this sequence is increasing. Consider, extending n to be a continuous variable,

$$\frac{d}{dn}(a_n) = \frac{d}{dn}\left[\frac{2n-3}{3n+4}\right]$$
$$= \frac{2(3n+4) - 3(2n-3)}{(3n+4)^2}$$
$$= \frac{17}{(3n+4)^2} > 0 \quad \text{for all } n \neq -4/3.$$

$\Rightarrow$ a_n is increasing on $[1, \infty)$

$\Rightarrow$ $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

We can find bounds from graphing $f(x) = \frac{2x-3}{3x+4}$, or we can just think about inequalities for a few moments,

$a_1 \leq a_2 \leq a_3 \leq \cdots$ since a_n is increasing. $\therefore$ it's monotonic

$-\frac{1}{7} \leq a_n$ for $n \geq 1$. (gives lower bound)

For upper bound notice $a_n = \frac{2n-3}{3n+4} = \frac{2}{3}\left[\frac{n - 3/2}{n + 4/3}\right] < \frac{2}{3}\left[\frac{n + 4/3}{n + 4/3}\right] = \frac{2}{3}$

made numerator bigger.

Hence $a_n \leq \frac{2}{3}$ for $n \geq 1$

$$\therefore \quad \boxed{-\frac{1}{7} \leq \frac{2n-3}{3n+4} \leq \frac{2}{3} \quad \text{for } n \in \mathbb{N}} \leftarrow \{a_n\} \text{ is bounded}$$

and again, yes it's monotonic.

$\boxed{\S 12.1 \# 64}$ Is $a_n = ne^{-n}$ monotonic and bounded?

I will use a completely CALC. I ⊕ graphing approach for this problem. I extend n to be a continuous variable and seek to graph $y = ne^{-n}$ via differential calculus' wisdom.

1.) The only zero is $n = 0$ and $(0,0)$ is on graph $y = a_n$.

2.) Critical #'s found from $\frac{da_n}{dn} = 0$

$$\frac{d}{dn}\left(ne^{-n}\right) = e^{-n} - ne^{-n} = (1-n)e^{-n} \quad \therefore \underbrace{n = 1}_{\text{only critical\#}}$$

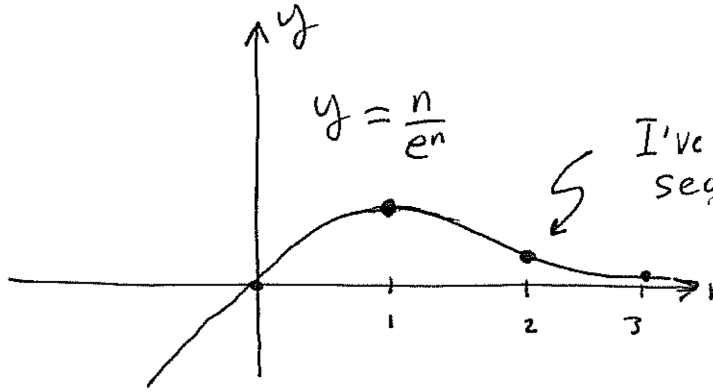
3.) $\overset{+ + + +}{} \underset{1}{|} \overset{- - - -}{\longrightarrow} \frac{da_n}{dn}$

By 1^{st} derivative test $(1, e^{-1})$ is a local max.

4.) $\lim\limits_{n \to -\infty}\left(ne^{-n}\right) = -\infty$ by common sense.

$\lim\limits_{n \to \infty}\left(ne^{-n}\right) = \lim\limits_{n \to \infty}\left(\frac{n}{e^n}\right) \overset{}{\underset{(\frac{\infty}{\infty})}{\overset{L}{=}}} \lim\limits_{n \to \infty}\left(\frac{1}{e^n}\right) = 0$

Hence the fnct has horizontal asymptote $y = 0$ as $n \to \infty$ whereas it diverges to $-\infty$ as $n \to -\infty$.



I've illustrated the actual sequence by the dots. From this graph we can deduce that

$\boxed{\begin{array}{l} \underline{a_n = ne^{-n} \text{ is decreasing}} \\ \text{and } 0 \le ne^{-n} \le 1/e \\ \text{so it's } \underline{\text{bounded}} \end{array}}$

(it's monotonic since it's decreasing)