Homework 2 : Calc. II : §7.7 # 9, 30, 31, 57, 58, 60, 63, 64 (Stew. Ed. 6)

§7.7 # 9 (kinda silly given my lecture)

$$\cosh x + \sinh x = \tfrac{1}{2}(e^x + \cancel{e^{-x}}) + \tfrac{1}{2}(e^x - \cancel{e^{-x}}) = \tfrac{1}{2}(e^x + e^x) = e^x.$$

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§7.7 # 30 Calculate the derivative of $f(x) = \tanh(1+e^{2x})$

Recall $\tanh(\theta) = \dfrac{\sinh\theta}{\cosh\theta}$. Also $(\cosh\theta)' = \sinh\theta$ & $(\sinh\theta)' = \cosh\theta$,

$$\frac{d}{d\theta}\left[\tanh\theta\right] = \frac{d}{d\theta}\left[\frac{\sinh\theta}{\cosh\theta}\right] = \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta} = \operatorname{sech}^2\theta$$

I used the well-known identity for $\cosh\theta$ & $\sinh\theta$. I'll derive it just to be complete here,

$$\begin{aligned}
\cosh^2\theta - \sinh^2\theta &= \left[\tfrac{1}{2}(e^\theta + e^{-\theta})\right]^2 - \left[\tfrac{1}{2}(e^\theta - e^{-\theta})\right]^2 \\
&= \tfrac{1}{4}\left[\cancel{e^{2\theta}} + 2 + \cancel{e^{-2\theta}}\right] - \tfrac{1}{4}\left[\cancel{e^{2\theta}} - 2 + \cancel{e^{-2\theta}}\right] \\
&= \tfrac{1}{4}\left[2 + 2\right] \\
&= 1.
\end{aligned}$$

Now differentiate,

$$\begin{aligned}
\frac{d}{dx}\left[\tanh(1+e^{2x})\right] &= \operatorname{sech}^2(1+e^{2x})\,\frac{d}{dx}(1+e^{2x}) \\
&= \operatorname{sech}^2(1+e^{2x}) \cdot 2e^{2x} \\
&= \boxed{2e^{2x}\operatorname{sech}^2(1+e^{2x})}
\end{aligned}$$

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§7.7 # 31 Again note $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$ and also $\frac{d}{dx}(\cosh x) = \sinh x$

$$\begin{aligned}
\frac{d}{dx}\left[x\sinh x - \cosh x\right] &= \sinh x + x\cosh x - \sinh x \\
&= \boxed{x\cosh x}
\end{aligned}$$

<u>Remark</u>: if you did this in exponential notation, sorry.

§7.7 #57

$$\int \sinh x \cosh^2 x \, dx = \int u^2 du \leftarrow \boxed{\begin{array}{l} u = \cosh x \\ du = \sinh x \, dx \end{array}}$$

$$= \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{3} \cosh^3(x) + C}$$

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§7.7 #58

$$\int \sinh(1 + 4x)\, dx = \int \sinh(u) \left(\tfrac{1}{4} du\right) \leftarrow \boxed{\begin{array}{c} u = 4x + 1 \\ du = 4dx \end{array}}$$

$$= \frac{1}{4} \cosh(u) + C$$

$$= \boxed{\frac{1}{4} \cosh(4x + 1) + C}$$

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§7.7 #60

$$\int \tanh(x)\, dx = \int \frac{\sinh(x)\, dx}{\cosh(x)}$$

$$= \int \frac{du}{u} \leftarrow \boxed{\begin{array}{l} u = \cosh x \\ du = \sinh x \, dx \end{array}}$$

$$= \ln|u| + C$$

$$= \ln|\cosh x| + C$$

note, $\cosh x = \frac{1}{2}(e^x + e^{-x}) \geq 1$ so I can drop $|\cdot|$ bars.

$$= \boxed{\ln(\cosh(x)) + C}$$

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§7.7 #63

$$\int_4^6 \frac{dt}{\sqrt{t^2 - 9}} = \int_4^6 \frac{dt}{3\sqrt{t^2/9 - 1}}$$

$$= \int_{4/3}^{2} \frac{3\, du}{3\sqrt{u^2 - 1}} \leftarrow \boxed{\begin{array}{l} \text{let } u = t/3, \; u^2 = t^2/9, \; 3du = dt \\ u(6) = 6/3 = 2, \; u(4) = 4/3. \end{array}}$$

$$= \left(\cosh^{-1}(2) - \cosh^{-1}(4/3)\right)$$

$$= \ln(2 + \sqrt{4 - 1}) - \ln\left(4/3 + \sqrt{16/9 - 1}\right)$$

see Eq $\stackrel{n}{=}$ 4 on pg. 466 (I'll derive it

$$= \ln(2 + \sqrt{3}) - \ln\left(\frac{4}{3} + \frac{1}{3}\sqrt{16 - 9}\right)$$

$$= \ln(2 + \sqrt{3}) - \ln(4 + \sqrt{7}) - \ln\left(\frac{1}{3}\right)$$

§7.7 #63 Continued

$$\int_4^6 \frac{dt}{\sqrt{t^2-9}} = \ln\left[\frac{(2+\sqrt{3})}{\frac{1}{3}(4+\sqrt{7})}\right] = \boxed{\ln\left[\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right]}$$

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The identity $\cosh^{-1}(x) = \ln(x - \sqrt{x^2-1})$ is derived as follows: let $\cosh^{-1}(x) = y \rightarrow \underbrace{\cosh(y) = x}$.

$$x = \tfrac{1}{2}(e^y + e^{-y})$$

$$2x = e^y + e^{-y}$$

$$\Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow \lambda^2 - 2x\lambda + 1 = 0, \quad \text{let } \lambda = e^y$$

$$\Rightarrow \lambda = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

choose (+) since $e^y > 0$.

$$\Rightarrow e^y = x + \sqrt{x^2-1}$$

$$\Rightarrow \cosh^{-1}(x) = y = \ln(e^y) = \ln(x + \sqrt{x^2-1})$$

$$\Rightarrow \boxed{\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})}$$

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§7.7 #64

$$\int_0^1 \frac{dt}{\sqrt{16t^2+1}} = \int_0^4 \frac{\frac{1}{4}\,du}{\sqrt{u^2+1}}$$

$\boxed{u = 4t, \quad du = 4dt, \quad u^2 = 16t^2, \quad u(0) = 0, \ u(1) = 4}$

$$= \frac{1}{4}\left[\sinh^{-1}(4) - \sinh^{-1}(0)\right]$$

$$= \boxed{\frac{1}{4}\sinh^{-1}(4)}$$

$$= \frac{1}{4}\ln(4 + \sqrt{16+1})$$

$$= \boxed{\frac{1}{4}\ln(4 + \sqrt{17})}$$

an equally nice answer given the right calculator.

Note $\sinh^{-1}(x) = y$

$\Rightarrow \sinh(y) = x$

$\Rightarrow \cosh y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$

$\Rightarrow \frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$

why $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + C$