HOMEWORK 23 : CALCULUS II : §12.9 # 4, 9, 10, 11, 17, 18, 23, 26, 28, 32 ①

(STEWART 6th Ed.)

§12.9 # 4 Find power series expansion for $f(x) = \frac{3}{1-x^4}$

$$f(x) = \frac{3}{1-x^4} = \sum_{n=0}^{\infty} 3(x^4)^n = \boxed{\sum_{n=0}^{\infty} 3x^{4n}} \quad \text{for } x \in (-1,1)$$

geom. series $a = 3$, $r = x^4$

I.O.C. = (-1,1)

since $r = x^4$ and $|r| < 1 \Rightarrow |x^4| < 1$
$\Rightarrow |x| < 1$.

§12.9 # 9 Find power series for $f(x) = \frac{1+x}{1-x}$

$$f(x) = \frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x}$$

: use geom. series result with $a = 1$ or $a = x$ and $r = x$.

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x(x)^n$$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1}$$

: let $k = n+1$ thus $n = 0 \Rightarrow k = 1$ and,

$$= 1 + \sum_{n=1}^{\infty} x^n + \sum_{k=1}^{\infty} x^k$$

$$= \boxed{1 + 2\sum_{n=1}^{\infty} x^n}$$

, the I.O.C. = (-1,1) by geometric series since $|r| < 1$ iff $|x| < 1$.

§12.9 #10 Find power series expansion of $f(x) = \frac{x^2}{a^3 - x^3}$

$$f(x) = \frac{x^2}{a^3(1 - x^3/a^3)} = \frac{x^2/a^3}{1 - x^3/a^3}$$

Identify $r = x^3/a^3$ & "a" $= x^2/a^3$ for geom. series,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{a^3}\left(\frac{x^3}{a^3}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}}}$$

where $\boxed{\text{I.O.C.} = (-|a|, |a|)}$

since $|r| < 1 \Rightarrow |x^3/a^3| < 1$
$\Rightarrow |x^3| < |a^3|$

$\boxed{\text{§12.9\#11}}$ Use partial fractions to help find power series expansion for $f(x) = \dfrac{3}{x^2 - x - 2}$

$$f(x) = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$\Rightarrow \quad 3 = A(x+1) + B(x-2)$$

$\underline{x=-1|}$ $\quad 3 = -3B \quad \therefore \underline{B=-1}$

$\underline{x=2|}$ $\quad 3 = 3A \quad \therefore \underline{A=1}$

Hence, $f(x) = \dfrac{-1}{x+1} + \dfrac{1}{x-2} = \dfrac{-1}{x+2} - \dfrac{1}{2(1 - x/2)}$

$$\therefore \quad f(x) = -\sum_{n=0}^{\infty} (-x)^n + \sum_{n=0}^{\infty} \frac{-1}{2}\left(\frac{x}{2}\right)^n$$ ← geom. series result.

$$= \boxed{\sum_{n=0}^{\infty} \left[(-1)^{n+1} - \left(\frac{1}{2}\right)^{n+1} \right] x^n} \qquad \text{I.O.C.} = \left(\tfrac{-1}{2}, \tfrac{1}{2}\right)$$

$\boxed{\text{§12.9\#17}}$ Find power series expansion for $f(x) = \dfrac{x^3}{(x-2)^2}$

Notice $g(x) = \dfrac{1}{(x-2)^2} \longrightarrow \int g(x)\,dx = \dfrac{-1}{x-2} \cancel{+C} = \dfrac{1}{2(1-x/2)} + C$

Hence, $\int g(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n \cancel{+C} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n + C$

$$\therefore \quad g(x) = \frac{d}{dx}\int g(x)\,dx = \sum_{n=0}^{\infty} \frac{n}{2^{n+1}} x^{n-1}$$

$$\Rightarrow \boxed{f(x) = x^3 g(x) = \sum_{n=0}^{\infty} \frac{n}{2^{n+1}} x^{n+2}}$$

(the R.O.C. is $R = 2$)

~~§12.9#18~~

$\boxed{\text{§12.9\#18}}$ Find power series for $f(x) = \tan^{-1}(x/3)$ and state the R.O.C. for the series

Notice $f'(x) = \dfrac{1/3}{1 + x^2/9} = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{x^2}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} x^{2n}$ then integrate to get back to $f(x)$,

$$f(x) = \int f'(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{2n+1}} x^{2n+1}$$: integrated term by term.

Note $f(0) = \tan^{-1}(0) = 0 = C$ thus,

$$\boxed{\tan^{-1}\left(\frac{x}{3}\right) = \sum \frac{(-1)^n}{2n+1}\left(\frac{1}{3^{2n+1}}\right) x^{2n+1}}$$

Remark: you can substitute $u = \frac{x}{3}$ into $\tan^{-1}(u)$ and get same result.

note the $\boxed{\text{R.O.C.} = 3}$ since $r < 1 \Rightarrow \left|\frac{x^2}{3^2}\right| < 1 \rightarrow |x| < 3 \rightarrow (-3,3) = \text{I.O.}$

§12.9 #23 Calculate a power series sol$^{\underline{n}}$ to $\int \frac{t}{1-t^8}\,dt$, what is the R.O.C.

$$\int\left(\frac{t}{1-t^8}\right)dt = \int\left(\sum_{n=0}^{\infty} t\,(t^8)^n\right)dt$$

$$= \int\left(\sum_{n=0}^{\infty} t^{8n+1}\right)dt$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}}$$

$\therefore$ let $a = t$, $r = t^8$

we need $|r| < 1$

hence $|t^8| < 1 \Rightarrow |t| < 1$

thus $\boxed{R.O.C. = 1}$

(integrating or differentiating will not change the R.O.C. I'm always using this fact in the problems of this section.)

§12.9 #26 Calculate $\int \tan^{-1}(x^2)\,dx$ as a power series and find that power series R.O.C.

Following #18, $f(x) = \tan^{-1}(x^2)$

from $a = 2x$, $r = -x^4$ $\Rightarrow |x| < 1$ $\therefore$ $\boxed{R.O.C. = 1}$

$$f'(x) = \frac{2x}{1+x^4} = \sum_{n=0}^{\infty} 2x(-x^4)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{4n+1}$$

$$f(x) = \int f'(x)\,dx = C + \sum_{n=0}^{\infty} \frac{2(-1)^n}{4n+2} x^{4n+2}$$

Note $f(0) = C = 0$ $\therefore$ $\tan^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}$.

Now I can do the integration,

$$\int \tan^{-1}(x^2)\,dx = \int\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}\right)dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} x^{4n+3}}$$

§12.9 #28 Calculate $\int_0^{0.4} \ln(1+x^4)\,dx$ to 6 decimal places

Notice $f(x) = \ln(1+x^4) \Rightarrow \frac{df}{dx} = \frac{4x^3}{1+x^4} = \sum_{n=0}^{\infty} 4x^3(-x^4)^n = \sum_{n=0}^{\infty} 4(-1)^n x^{4n+3}$

Then $f(x) = \int f'(x)\,dx = c + \sum_{n=0}^{\infty} \frac{4(-1)^n}{4n+4} x^{4n+4}$

Observe $f(0) = \ln(1) = 0 = c \quad \therefore f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{4n+4}$

Thus we can expand the integrand. I'll write the first few terms because I anticipate we'll have an alternating series so I can use the $|S - S_n| \leq b_{n+1}$ error Th$^{\underline{m}}$,

$$\int_0^{0.4} \ln(1+x^4)\,dx = \int_0^{0.4} \left(x^4 - \tfrac{1}{2}x^8 + \tfrac{1}{3}x^{12} - \tfrac{1}{4}x^{16} + \cdots\right)dx$$

$$= \left[\tfrac{1}{5}x^5 - \tfrac{1}{18}x^9 + \tfrac{1}{39}x^{13} - \tfrac{1}{56}x^{17} + \cdots\right]\Big|_0^{0.4}$$

$$= \underbrace{\tfrac{1}{5}(0.4)^5 - \tfrac{1}{18}(0.4)^9 + \tfrac{1}{39}(0.4)^{13}}_{\text{these certainly give integral to 6 decimals}} - \cdots = \boxed{0.002034}$$

since $\left|\frac{1}{56}(0.4)^{17}\right| = \underbrace{\frac{4^{17}}{56}}_{3.07\times10^{-9}}\left(\frac{1}{10}\right)^{17} < 0.0000001$.

(keeping 3 terms gets 8 decimals.)

Remark: $b_3 = 1.72\times10^{-7}$ thus $\frac{1}{5}(0.4)^5 - \frac{1}{18}(0.4)^9$ is enough in fact.

§12.9 #32 Show that $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ is a sol$^{\underline{n}}$ to $y'' + y = 0$

Notice that

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{(2n)!} x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1}$$

: note $2n = 0$ when $n = 0$ so we drop $n = 0$.

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}(2n-1)x^{2n-2}$$

: notice we keep $n = 1$ since $2n-1 = 2-1 = 1 \neq 0$ for the lowest term.

Let's change the index of the $f''(x)$ sum. Also notice $\frac{2n-1}{(2n-1)!} = \frac{1}{(2n-2)!}$

$$f''(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k}$$

Let $2k = 2n - 2$
$n = 1 \Rightarrow 2k = 2 - 2 = 0 \Rightarrow k = 0$
$\underline{n = k + 1}$.

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= -f(x) \quad \therefore \quad f''(x) + f(x) = 0 \quad \therefore \quad f(x) \text{ solves } y'' + y = 0$$