FOURIER Add-on # 1 Let $f(x) = \begin{cases} 1 & \text{if } -\pi \le x < 0 \\ -1 & \text{if } 0 \le x < \pi \end{cases}$ and suppose f is periodic with period 2π.

(a.) find Fourier Coefficients

(b.) find Fourier series of f. For what values of x is $f(x)$ = Fourier Series generated by $f(x)$?

(c.) graph $y = S_2$, $y = S_4$, $y = S_6$ on same graph.

(a.)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} dx - \int_{0}^{\pi} dx\right) = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx - \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} - \frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right]$$

$= 0$ (as we could have expected since $f(x)\cos(nx)$ is an odd function integrated symmetrically about a finite interval at zero.)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

: note $f(x)\sin(nx)$ is even thus we can just integrate from $[0, \pi]$ then double it.

$$= \frac{2}{\pi}\int_{0}^{\pi} -\sin(nx)\,dx$$

$$= \frac{2}{\pi}\;\frac{\cos(nx)}{n}\Big|_{0}^{\pi}$$

$$= \frac{2}{\pi n}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \frac{2}{n\pi}\left((-1)^n - 1\right) \qquad n = 1, 2, 3, \ldots$$

Thus $\boxed{a_n = 0 \text{ and } b_n = \frac{2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} -4/n\pi & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}}$

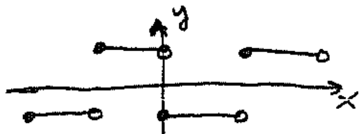
(b.)

$$f(x) \sim \sum_{k=0}^{\infty} \frac{-4}{(2k+1)\pi}\sin[(2k+1)x]$$

(will disagree with $f(x)$ at discontinuities)

$$f(x) \sim -\frac{4}{\pi}\sin(x) - \frac{4}{3\pi}\sin(3x) - \frac{4}{5\pi}\sin(5x) - \frac{4}{7\pi}\sin(7x) - \ldots$$

(c.)



$y = f(x)$

(partial answer I leave graphs

the Fourier series above seems reasonable, it's positive where $f(x)$ is positive roughly speaking...

$\boxed{f(x) \ne \text{Fourier}(x) \text{ for } x = n\pi,\ n \in \mathbb{Z}.}$

<u>Remark</u>: the notation $f(x) \sim g(x)$ means that $f(x) = g(x)$ for <u>most</u> x on some domain. Usually $f(x) \neq g(x)$ only for some <u>finite</u> subset of dom(f). For the Fourier series the disagreements will occur at the jump discontinuities. In problem #1 we had $f(x) \neq$ Fourier(x) for $x = 0, \pm\pi, \pm 2\pi, \ldots$

Fourier Add-on #4 Same instructions as #1 but with $f(x) = x^2$

Notice $f(x) = x^2$ is an <u>even</u> function. Thus, $f(x)\cos(nx)$ is <u>even</u> whereas $f(x)\sin(nx)$ is odd. It follows that $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx = 0$, for $n = 1, 2, \ldots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{6\pi}x^3\Big|_{-\pi}^{\pi} = \frac{1}{6\pi}\left(\pi^3 - (-\pi)^3\right) = \underline{\frac{\pi^2}{3} = a_0}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{\underset{\sim}{x^2}}\,\underbrace{\cos(nx)\,dx}_{dV} \quad : \quad \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \qquad \begin{matrix} dV = \cos(nx)\,dx \\ V = \frac{1}{n}\sin(nx) \end{matrix}$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x^2\sin(nx)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)\cdot 2x\,dx\right]$$

$$= \frac{-4}{n\pi}\int_0^{\pi} \underset{u}{\underset{\sim}{x}}\,\underbrace{\sin(nx)\,dx}_{dV} \quad : \quad \begin{matrix} u = x \\ du = dx \end{matrix} \qquad \begin{matrix} dV = \sin(nx)\,dx \\ V = -\frac{1}{n}\cos(nx) \end{matrix}$$

$$= \frac{-4}{n\pi}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right]$$

$$= \frac{-4}{n\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \cancel{\frac{1}{n^2}\sin(nx)\Big|_0^{\pi}}^{\,0}\right]$$

$$= \frac{4}{n^2}\cos(n\pi)$$

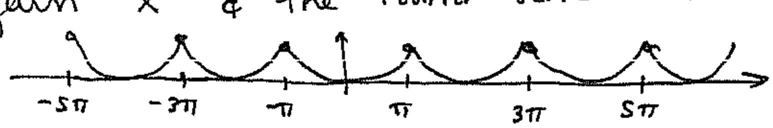
$$\therefore \quad \boxed{a_0 = \pi^2/3, \quad a_n = \frac{4}{n^2}(-1)^n \text{ for } n = 1, 2, \ldots \quad b_n = 0 \text{ for } n = 1, 2, \ldots}$$

Thus,

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \quad (nx)$$

$$\sim \frac{\pi^2}{3} - 4\sin(x) + \sin(2x) - \frac{4}{9}\sin(3x) + \frac{4}{16}\sin(4x) - \cdots$$

Again x^2 & the Fourier series will not match at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$



in contrast to #1 no problem at $x = \pm 2k\pi$ for $k = 0, 1, 2, \ldots$

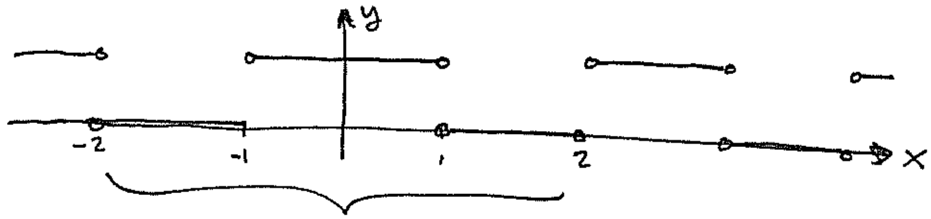
Fourier Add-on # 7

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } 1 \le |x| < 2 \end{cases} \qquad \text{where} \quad \underbrace{f(x+4) = f(x)}_{\text{period of } 4 = 2L}$$

$\therefore \ \underline{L = 2}$

Find Fourier series for $f(x)$

We propose $f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$. Following $eq^{\underline{n}}$ box # 9, before that let's graph $f(x)$ to check on my claim for L,



$y = f(x)$.

period $= 4 = 2L$
$\therefore \ L = 2.$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}(2) = \underline{\frac{1}{2} = a_0}.$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} \cos\left(\frac{n\pi x}{2}\right)dx = \int_{0}^{1} \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$

$$= \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(0)\right)$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

In contrast $b_n = 0$ for $n = 1, 2, \ldots$ since $f(x)\sin\left(\frac{n\pi x}{2}\right)$ is odd.

Therefore,

$$\boxed{\begin{aligned} f(x) &\sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right) \\ &\sim \frac{1}{2} + \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right) + \frac{2}{5\pi}\cos\left(\frac{5\pi x}{2}\right) - \frac{2}{7\pi}\cos\left(\frac{7\pi x}{2}\right) + \cdots \end{aligned}}$$

Fourier Add-on #11

Find Fourier Series for $f(t) = \sin(3\pi t)$ periodic over $-1 \le t \le 1$

I identify $L = 1$. Also, $f(t)$ & $f(t)\cos(n\pi t)$ are odd $\therefore$ $a_n = 0$ for $n = 0, 1, 2, \ldots$

Calculate

$$b_n = \frac{1}{1}\int_{-1}^{1} \sin(3\pi t)\sin(n\pi t)\,dt$$

Notice $\sin(3\pi t)\sin(n\pi t) = \frac{1}{2}\cos((3-n)\pi t) - \frac{1}{2}\cos((3+n)\pi t)$

Thus for $n \neq 3$,

$$\begin{aligned} b_n &= \int_{-1}^{1}\left[\frac{1}{2}\cos((3-n)\pi t) - \frac{1}{2}\cos((3+n)\pi t)\right]dt \\ &= \left[\frac{1}{2}\frac{1}{(3-n)\pi}\sin((3-n)\pi t) - \frac{1}{2}\frac{1}{(3+n)\pi}\sin((3+n)\pi t)\right]\Bigg|_{-1}^{1} \\ &= 0 \quad \text{since } \sin(k\pi) = 0 \text{ for all } k \in \mathbb{Z}. \end{aligned}$$

In contrast, $b_3 \neq 0$ since

$$\begin{aligned} b_3 &= \int_{-1}^{1} \sin(3\pi t)\sin(3\pi t)\,dt \\ &= \int_{-1}^{1}\left[\frac{1}{2} - \frac{1}{2}\cos(6\pi t)\right]dt \\ &= \left[\frac{t}{2} - \frac{1}{12\pi}\sin(6\pi t)\right]\Big|_{-1}^{1} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1. \end{aligned}$$

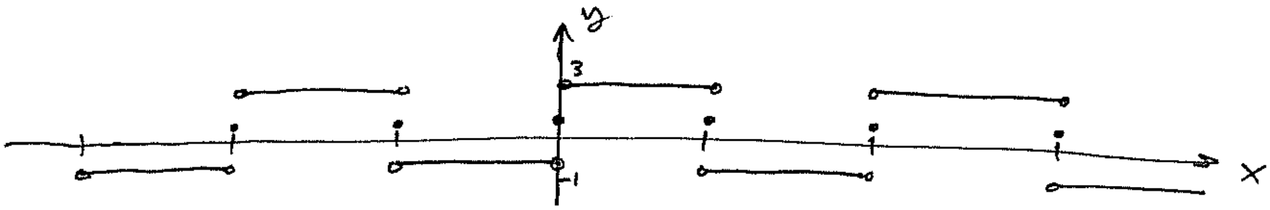
$\therefore$ $\boxed{b_3 = 1}$ $\quad$ $a_m = 0$, $m = 0, 1, 2, \ldots$, $\quad$ $b_n = 0$, $n \neq 3$

Hence, $f(t) = \sin(3\pi t)$ is the Fourier Series for $f(t)$

(Ha.)

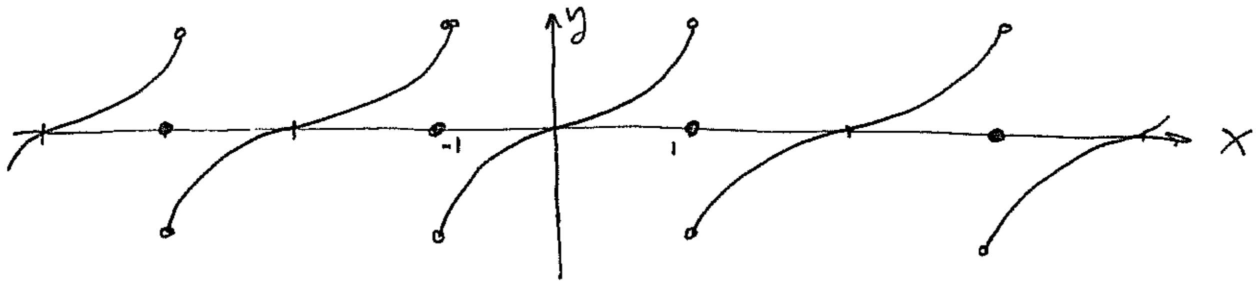
(this is analogous to asking for Taylor Series about zero for a polynomial.)

Fourier #13 Sketch Fourier expansion's graph for $f(x) = \begin{cases} -1 & -4 \le x < 0 \\ 3 & 0 \le x \le 4 \end{cases}$



At the jumps the Fourier Series goes to the average value of $\underline{y = 1}$.

Fourier #15 Sketch graph of Fourier expansion of $f(x) = x^3$, $-1 \le x \le 1$



Fourier #17

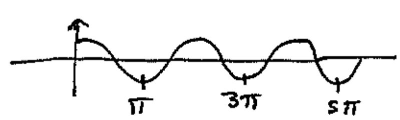
(a.) Show $x^2 \sim \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$ for $-1 \le x \le 1$.

(b.) By substituting a particular x-value show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(a.) By almost the same calculations as #4 we can show

$$a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}, \quad a_n = \frac{4}{n^2\pi^2}(-1)^n, \quad b_n = 0$$

(b.) We want $\cos(n\pi x) = (-1)^n$



Let $x = 1$,

$$\cos(n\pi) = \begin{cases} 1 & \text{if } n = 2, 4, 0, -2, \ldots \\ -1 & \text{if } n = 1, 3, -1, -3, \ldots \end{cases} \quad \therefore \cos(n\pi) = (-1)^n$$

Thus $1^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2\pi^2} (-1)^n$

$$\Rightarrow \frac{2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

(!!!)