§8.3 #1 Use $x = 3\sec\theta$ to evaluate the $\int$ below.

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}} \quad : \quad \begin{array}{l} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\,d\theta \end{array}$$

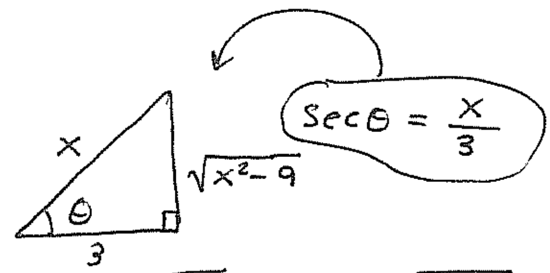
$$= \int \frac{\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\tan\theta} \quad : \quad \left[\begin{array}{l} 9\sec^2\theta - 9 = 9\tan^2\theta. \\ \Rightarrow \sqrt{9\sec^2\theta-9} = 3\tan\theta. \end{array}\right]$$

$$= \int \frac{d\theta}{9\sec\theta}$$

$$= \frac{1}{9}\int \cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \boxed{\frac{1}{9}\left(\frac{\sqrt{x^2-9}}{x}\right) + C}.$$



$$\sin\theta = \frac{OPP}{HYP} = \frac{\sqrt{x^2-9}}{x}$$

§8.3 #2 Use $x = 3\sin\theta$ to evaluate $\int$ below.

$$\int x^3\sqrt{9-x^2}\,dx = \int 27\sin^3\theta\sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta \quad : \quad \left[\begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{array}\right]$$

$$= \int 81\sqrt{9}\,\sin^3\theta\cos^2\theta\,d\theta$$

$$= 243\int \sin^2\theta\cos^2\theta\sin\theta\,d\theta$$

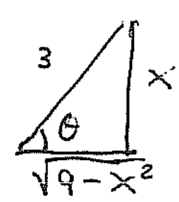
$$= 243\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$$

$$= 243\int (1-u^2)u^2(-du) \quad : \quad \left[\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right]$$

$$= 243\int (u^4-u^2)\,du$$

$$= 243\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= 243\left(\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right) + C$$



$$= \frac{243}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 - \frac{243}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C$$

$243 = 3^5$

$$= \boxed{\frac{1}{5}\left(\sqrt{9-x^2}\right)^5 - 3\left(\sqrt{9-x^2}\right)^3 + C}$$

§8.3 #3 use $x = 3\tan\theta$,

$$\int \frac{x^3\,dx}{\sqrt{x^2+9}} = \int \frac{27\tan^3\theta \; 3\sec^2\theta\,d\theta}{\sqrt{9\tan^2\theta+9}} \qquad \left[\begin{array}{l} x = 3\tan\theta \\ dx = 3\sec^2\theta\,d\theta \end{array}\right]$$

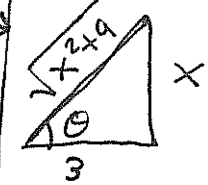
$$= \int \frac{81\tan^3\theta\,\sec^2\theta\,d\theta}{3\sec\theta}$$

$$= 27\int(-1+\sec^2\theta)\sec\theta\tan\theta\,d\theta \qquad : \left[-1+\sec^2\theta = \tan^2\theta\right]$$

$$= 27\int(u^2-1)\,du \qquad : \left[\begin{array}{l} u = \sec\theta \\ du = \sec\theta\tan\theta\,d\theta \end{array}\right]$$

$$= 27\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$

$$= \boxed{9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + C}$$



$$\tan\theta = \frac{x}{3} \qquad \sec\theta = \frac{\sqrt{x^2+9}}{3}$$

[perfectly good answer.]

$$= \frac{1}{3}\sqrt{x^2+9}\,(x^2+9) - 9\sqrt{x^2+9} + C$$

$$= \sqrt{x^2+9}\left[\frac{1}{3}(x^2+9) - 9\right] + C$$

$$= \boxed{\frac{1}{3}\sqrt{x^2+9}\left[x^2-18\right] + C.}$$

(just showing how to get texts answer)

§8.3#10

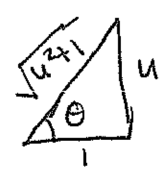
$$\int \frac{t^5\,dt}{\sqrt{t^2+2}} = \frac{1}{\sqrt{2}}\int \frac{t^5\,dt}{\sqrt{t^2/2+1}} \qquad : \text{let } u = t/\sqrt{2} \to t = u\sqrt{2}, \text{ then } du = \frac{1}{\sqrt{2}}\,dt$$

$$= \frac{(\sqrt{2})^5}{\sqrt{2}}\int \frac{u^5\sqrt{2}\,du}{\sqrt{u^2+1}}$$

$$= 4\sqrt{2}\int \frac{u^5\,du}{\sqrt{u^2+1}} \qquad : \left[\begin{array}{l} \text{let } u = \tan\theta, \quad du = \sec^2\theta\,d\theta \\ u^2+1 = \tan^2\theta+1 = \sec^2\theta \end{array}\right]$$

$$= 4\sqrt{2}\int \frac{\tan^5\theta\,\sec^2\theta\,d\theta}{\sec\theta}$$

$$= 4\sqrt{2}\int(\sec^2\theta-1)\sec\theta\tan\theta\,d\theta$$



$$\tan\theta = \frac{u}{1}$$

$$= 4\sqrt{2}\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 4\sqrt{2}\left(\frac{1}{3}\left(\sqrt{u^2+1}\right)^3 - \sqrt{u^2+1}\right) + C = 4\sqrt{2}\left(\frac{1}{3}\left(\sqrt{t^2/2+1}\right)^3 - \sqrt{t^2/2+1}\right) +$$

ok

§ 8.3 #10 Cleaning up answer,

$$\int \frac{t^5\,dt}{\sqrt{t^2+2}} = \left[\frac{4\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}\sqrt{t^2+2}\right)^3 - \frac{3}{\sqrt{2}}\sqrt{t^2+2}\right] + C$$

$$= \frac{4}{3}\left[(t^2+2)\frac{1}{2} - 3\right]\sqrt{t^2+2} + C$$

$$= \frac{4}{3}\left[\frac{1}{2}t^2 - 2\right]\sqrt{t^2+2} + C$$

$$= \boxed{\frac{2}{3}(t^2-4)\sqrt{t^2+2} + C}$$

better answer.

(assuming I made no arithmetic mistake... gulp.)

§ 8.3 # 18

$$\int \frac{dx}{[(ax)^2-b^2]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\tan\theta\,d\theta}{b^3\tan^3\theta}$$

: let $ax = b\sec\theta$

$(ax)^2 - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$

$[(ax)^2-b^2]^{3/2} = b^3\tan^3\theta.$

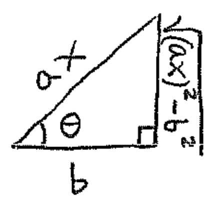
$dx = \frac{b}{a}\sec\theta\tan\theta\,d\theta$

$$= \frac{1}{ab^2}\int \frac{\sec\theta\,d\theta}{\tan^2\theta}$$

$$= \frac{1}{ab^2}\int \frac{\cos^2\theta\,d\theta}{\cos\theta\sin^2\theta}$$

$$= \frac{1}{ab^2}\int \frac{\cos\theta\,d\theta}{\sin^2\theta}$$

$$= \frac{1}{ab^2}\left(\frac{-1}{\sin\theta}\right) + C$$



$\sec\theta = \frac{ax}{b}$

$$= \frac{-ax}{ab^2\sqrt{(ax)^2-b^2}} + C$$

$$= \boxed{\frac{-x}{b^2\sqrt{(ax)^2-b^2}} + C.}$$

§8.3 #32)

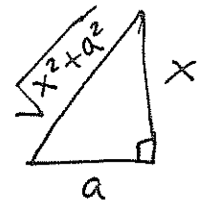
$$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = \int \frac{a^2\tan^2\theta \; a\sec^2\theta\,d\theta}{(a^2\sec^2\theta)^{3/2}} \quad : \quad \left[\begin{array}{l} \text{Let } x = a\tan\theta \\ x^2+a^2 = a^2\sec^2\theta \\ dx = a\sec^2\theta\,d\theta \end{array}\right.$$

$$= \int \frac{a^3\tan^2\theta\,d\theta}{a^3\sec\theta}$$

$$= \int \left(\frac{\sec^2\theta - 1}{\sec\theta}\right) d\theta$$

$$= \int \left(\sec\theta - \frac{1}{\sec\theta}\right) d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta$$



$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \boxed{\ln\left|\sqrt{x^2+a^2} + x\right| - \frac{x}{\sqrt{x^2+a^2}} + C_2}$$

Same $\int$, but by hyperbolic trig. subst. $\underline{x = a\sinh t}$.

$$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = \int \frac{(a^2\sinh^2 t)(a\cosh t\,dt)}{a^3\cosh^3 t} \quad \left[\begin{array}{l} x^2+a^2 = a^2\cosh^2 t \\ dx = a\cosh t\,dt \end{array}\right]$$

$$= \int \left(\frac{\sinh^2 t}{\cosh^2 t}\right) dt \quad : \quad \cosh^2 t - \sinh^2 t = 1 \;\rightarrow\; \underline{\sinh^2 t = \cosh^2 t - 1}$$

$$= \int \left(\frac{\cosh^2 t - 1}{\cosh^2 t}\right) dt$$

$$= \int (1 - \operatorname{sech}^2 t)\,dt$$

$$= t - \tanh t + C \longleftarrow$$

See pg. 465 of text

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

$$= \boxed{\sinh^{-1}\left(\frac{x}{a}\right) - \tanh\left(\sinh^{-1}\left(\frac{x}{a}\right)\right) + C}$$

§8.3 #24

$$\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{dt}{\sqrt{(t-3)^2+4}}$$

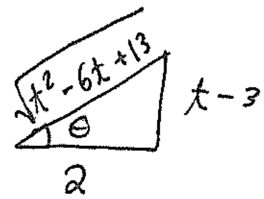
$$\left[\begin{aligned} t-3 &= 2\tan\theta \\ (t-3)^2+4 &= 4\tan^2\theta+4 = 4\sec^2\theta \\ dt &= 2\sec^2\theta\, d\theta. \end{aligned}\right.$$

$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \sec\theta\, d\theta \qquad : \left[\text{since } \frac{du}{u} = \sec\theta\, d\theta \text{ for } u = \sec\theta+\tan\theta\right]$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{t^2-6t+13}}{2} + \frac{t-3}{2}\right| + C$$



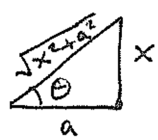
$$= \boxed{\ln\left|\sqrt{t^2-6t+13} + t-3\right| + C_2}$$

§8.3 #31

(a.) $$\int \frac{dx}{\underbrace{\sqrt{x^2+a^2}}} = \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta| + C$$

$x = a\tan\theta$

$x^2+a^2 = a^2\sec^2\theta$



$$= \ln\left|\frac{1}{a}\sqrt{x^2+a^2} + \frac{x}{a}\right| + C$$

$$= \boxed{\ln\left|\sqrt{x^2+a^2} + x\right| + C_2}$$

(b.) Let $x = a\sinh t$ so $dx = a\cosh t\, dt$
and $x^2+a^2 = a^2(1+\sinh^2 t) = a^2\cosh^2 t$.

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\, dt}{a\cosh t}$$

$$= t + C$$

$$= \boxed{\sinh^{-1}\left(\frac{x}{a}\right) + C}$$

Remark: in §7.7 we can prove

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

Replace $x \longmapsto \frac{x}{a}$ to connect our integrations in #31

§8.3 # 38

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b\, dx}{4\pi\epsilon_0 (x^2+b^2)^{3/2}}$$

Let's solve the integral w/o bounds then return to the given problem.

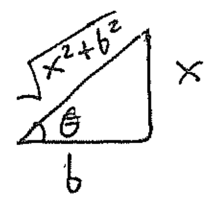
$$\int \frac{dx}{(x^2+b^2)^{3/2}} = \int \frac{b\sec^2\theta\, d\theta}{(b^2\sec^2\theta)^{3/2}}$$

$$\left[\begin{array}{l} x = b\tan\theta \\ x^2+b^2 = b^2\sec^2\theta \\ dx = b\sec^2\theta\, d\theta \end{array}\right]$$

$$= \int \frac{b\sec^2\theta\, d\theta}{b^3\sec^3\theta}$$

$$= \frac{1}{b^2}\int \cos\theta\, d\theta$$

$$= \frac{-1}{b^2}\sin\theta + C$$



$$= \boxed{\frac{-x}{b^2\sqrt{x^2+b^2}} + C}$$ I

It follows that

$$E(P) = \frac{\lambda b}{4\pi\epsilon_0}\int_{-a}^{L-a} \frac{dx}{(x^2+b^2)^{3/2}}$$

$$= \frac{\lambda b}{4\pi\epsilon_0}\left[\frac{-x}{b^2\sqrt{x^2+b^2}}\right]_{-a}^{L-a}$$

$$= \boxed{\frac{\lambda}{4\pi\epsilon_0 b}\left[\frac{-(L-a)}{\sqrt{(L-a)^2+b^2}} - \frac{a}{\sqrt{a^2+b^2}}\right]}$$