Homework 7 : §8.4 # 2, 4, 6, 7, 8, 10, 13, 19, 20, 29, 34, 35, 48, 51, 69 : Calculus II

(Stewart 6th Ed)

§8.4#2

a.) $$\frac{x}{x^2+x-2} = \frac{x}{(x+\frac{1}{2})(x-1)} = \boxed{\frac{A}{x+2} + \frac{B}{x-1}}$$

b.) $$\frac{x^2}{x^2+x-2} = \frac{x^2+x-2-x+2}{x^2+x-2} = 1 + \frac{2-x}{x^2+x-2} = \boxed{1 + \frac{A}{x+2} + \frac{B}{x-1}}$$

§8.4#4

a.) $$\frac{x^3}{x^2+4x+3}$$

$$\begin{array}{r l} & \;\; x - 4 \\ x^2+4x+3 \,\big) & x^3 \\ & x^3+4x^2+3x \\ \hline & -4x^2-3x \\ & -4x^2-16x-12 \\ \hline & \boxed{13x+12} \end{array}$$

$$\to \frac{x^3}{x^2+4x+3} = x - 4 + \frac{13x+12}{x^2+4x+3}$$

$$\Rightarrow \frac{x^3}{x^2+4x+3} = \boxed{x - 4 + \frac{A}{x+1} + \frac{B}{x+3}}.$$

(b.) $$\frac{2x+1}{(x+1)^3(x^2+4)^2} = \boxed{\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}}.$$

§8.4#6

(a.) $$\frac{x^4}{(x^3+x)(x^2-x+3)} = \frac{x^3}{(x^2+1)(x^2-x+3)} = \boxed{\frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2-x+3)}}.$$

(b.) $$\frac{1}{x^6-x^3} = \frac{1}{x^3}\left(\frac{1}{x^3-1}\right) = \frac{1}{x^3(x-1)(x^2+x+1)} =$$

notice $x=1$ makes this zero

$$\begin{array}{r l} & \;\; x^2+x+1 \\ x-1 \,\big) & x^3-1 \\ & x^3-x^2 \\ \hline & x^2-1 \\ & x^2-x \\ \hline & x-1 \\ & x-1 \end{array}$$

$$= \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+x+1}}$$

Remark: this one is neat.

(2)

§ 8.4 # 7

$$\int \frac{x}{x-6}\,dx = \int \frac{x-6+6}{x-6}\,dx = \int\left(1+\frac{6}{x-6}\right)dx = \boxed{x + 6\ln|x-6| + C}$$

§ 8.4 # 8

$$\int \underbrace{\frac{r^2}{r+4}}_{\text{improper fraction}} dr = \int\left(r - 4 + \frac{16}{r+4}\right)dr = \boxed{\frac{1}{2}r^2 - 4r + 16\ln|r+4| + C}$$

$$\begin{array}{r} r-4 \\ r+4\,\overline{)\,r^2\qquad} \\ \underline{r^2+4r} \\ -4r\qquad \\ \underline{-4r-16} \\ \boxed{16.} \end{array}$$

§ 8.4 # 10

$$\int \frac{dt}{(t+4)(t-1)} = \int \frac{-\,dt}{5(t+4)} + \int \frac{dt}{5(t-1)}$$

$$= \boxed{-\frac{1}{5}\ln|t+4| + \frac{1}{5}\ln|t-1| + C}$$

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1}$$

$$1 = A(t-1) + B(t+4)$$

$\underline{t=1}$ $\quad 1 = 5B \quad \therefore \quad B = 1/5$

$\underline{t=-4}$ $\quad 1 = -5A \quad \therefore \quad A = -1/5$

§ 8.4 # 13

$$\int \frac{ax\,dx}{x^2 - bx} = \int \underbrace{\frac{a\,dx}{x-b}}_{\substack{\text{let } u = x-b \\ du = dx}} = \int \frac{a\,du}{u} = a\ln|u| + C = \boxed{a\ln|x-b| + C}$$

§ 8.4 # 19

$$\int \frac{dx}{(x+5)^2(x-1)} = \frac{-1}{36}\int \frac{dx}{x+5} - \frac{1}{6}\int \frac{dx}{(x+5)^2} + \frac{1}{36}\int \frac{dx}{x-1}$$

$$= \boxed{\frac{-1}{36}\ln|x+5| + \frac{1}{6}\left(\frac{1}{x+5}\right) + \frac{1}{36}\ln|x-1| + C}$$

$$\frac{1}{(x+5)^2(x-1)} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1}$$

$$1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$$

$\underline{x=-5}$ $\quad 1 = -6B \quad \therefore \quad B = -1/6$

$\underline{x=1}$ $\quad 1 = 36C \quad \therefore \quad C = 1/36$

$\underline{x=0}$ $\quad 1 = -5A - B + 25C$

$$5A = -1 - B + 25C$$

$$A = \frac{1}{5}\left[-1 + \frac{1}{6} + \frac{25}{36}\right]$$

$$= \frac{1}{5}\left[\frac{-36+6+25}{36}\right]$$

$$= \frac{-1}{36} = A.$$

§8.4#20

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2-5x+16 = A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$$

$x=2$ $\quad 4-10+16 = 5C \Rightarrow 10 = 5C \Rightarrow \underline{C=2}$.

$x=-\frac{1}{2}$

$$\frac{1}{4}+\frac{5}{2}+16 = \frac{1+10+64}{4} = \frac{75}{4} = A\left(\frac{-1}{2}-2\right)^2, \quad \frac{-1}{2}-2 = \frac{-5}{2}$$

$$\frac{75}{4} = A\left(\frac{25}{4}\right) \Rightarrow \underline{A=3}.$$

$x=0$ $\quad 16 = 4A-2B+C$

$$2B = 4A+C-16$$
$$B = 2A+\frac{1}{2}C-8$$
$$B = 6+1-8 \Rightarrow \underline{B=-1}.$$

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx = \int\left[\frac{3}{2x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}\right]dx$$

$$= \boxed{\frac{3}{2}\ln|2x+1| - \ln|x-2| - 2\left(\frac{1}{x-2}\right) + C}$$

§8.4#29

$$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1+3}{(x+1)^2+4}\,dx$$

$$= \underbrace{\int \frac{(x+1)\,dx}{(x+1)^2+4}}_{\substack{u=(x+1)^2+4 \\ du=2(x+1)\,dx}} + \underbrace{\int \frac{3\,dx}{(x+1)^2+4}}_{\substack{x+1=2\tan\theta,\ dx=2\sec^2\theta\,d\theta \\ (x+1)^2+4=4\tan^2\theta+4=4\sec^2\theta}}$$

$$= \int \frac{\frac{1}{2}du}{u} + \int \frac{6\sec^2\theta\,d\theta}{4\sec^2\theta}$$

$$= \frac{1}{2}\ln|u| + \frac{3}{2}\theta + C$$

note $\tan\theta = \frac{x+1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{x+1}{2}\right)$.

$$= \boxed{\frac{1}{2}\ln|x^2+2x+5| + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C}$$

§8.4 # 34

$$\begin{aligned}
\int \frac{x^3}{x^3+1}\,dx &= \int \frac{x^3+1\ -1}{x^3+1}\,dx \\
&= \int \left(1 - \frac{1}{x^3+1}\right)dx \\
&= x - \int \frac{dx}{x^3+1}
\end{aligned}$$

Notice $(-1)^3+1 = 0$ thus $x+1$ divides x^3+1.

$$\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x+1 & x^3 + 1 \\
 & x^2 + x^2 \\
 & -x^2 + 1 \\
 & -x^2 - x \\
 & x + 1 \\
 & x + 1 \\
 & 0
\end{array}$$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$$

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$\Rightarrow 1 = x^2(A+B) + x(-A+B+C) + A + C$$

Equating Coefficients,

$$\begin{aligned}
x^2)&\quad 0 = A+B \\
x)&\quad 0 = -A+B+C \\
1)&\quad 1 = A+C
\end{aligned}$$

Put in $x = -1$

$1 = 3A$

$A = 1/3$.

$\Rightarrow B = -1/3$

$\Rightarrow C = 2/3$.

Thus,

$$\begin{aligned}
\int \frac{x^3}{x^3+1}\,dx &= x - \int\left[\frac{1/3}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}\right]dx \\
&= x - \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\left(\frac{x-2}{x^2-x+1}\right)dx \\
&= x - \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx + \frac{1}{3}\underbrace{\int\frac{-\frac{3}{2}\,dx}{(x-\frac{1}{2})^2+\frac{3}{4}}} \\
&= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln\left|(x-\tfrac{1}{2})^2+\tfrac{3}{4}\right| - \frac{1}{2}\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta} \\
&= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C \\
&= \boxed{x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C}
\end{aligned}$$

$x - \frac{1}{2} = a\tan\theta$

$a = \sqrt{3}/2$

§8.4 # 35

$$\int \frac{dx}{x(x^2+4)^2} \qquad \text{note} \qquad \frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$$

$$1 = A(x^4+8x^2+16) + B(x^4+4x^2) + C(x^3+4x) + Dx^2 + Ex$$

$$1 = x^4[A+B] + x^3[C] + x^2[8A+4B+D] + x[4C+E] + 16A$$

Equate Coefficients,

x^0 | $16A = 1 \Rightarrow \underline{A = 1/16}$.

x | $0 = 4C+E$

x^2 | $0 = 8A+4B+D$

x^3 | $\underline{0 = C}$. $\Rightarrow \underline{E = 0}$. (given $4C+E=0$)

x^4 | $0 = A+B \Rightarrow \underline{B = -1/16}$.

Finally $D = -8A-4B = \frac{-8}{16} + \frac{4}{16} = \frac{-4}{16} = \underline{\frac{-1}{4} = D}$.

$$\int \frac{dx}{x(x^2+4)^2} = \frac{1}{16}\int \frac{dx}{x} - \frac{1}{16}\int \frac{x\,dx}{x^2+4} - \frac{1}{4}\int \frac{x\,dx}{(x^2+4)^2}$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+4| + \frac{1}{8}\left(\frac{1}{(x^2+4)}\right) + C$$

$$= \boxed{\frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C}$$

awww... STEWERT WAS NICE TO YOU GUYS. all u-subst.

(just trying to write neater nothing profound here.)

§ 8.4 # 48 |

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u}$$

$$\frac{1}{u^2+u} = \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

$\underline{u=-1|}$ $\quad 1 = -B \quad \therefore \quad \underline{B=-1}$.

$\underline{u=0|}$ $\quad \underline{1 = A}$.

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u} - \int \frac{du}{u+1}$$

$$= \ln|u| - \ln|u+1| + C$$

$$= \boxed{\ln|\sin x| - \ln|\sin x + 1| + C}$$

$$= -\ln\left|\frac{\sin x + 1}{\sin x}\right| + C$$

$$= \underline{-\ln|1 + \csc(x)| + C}.$$

you could play games with making a single ln term.

of course there are dozens of other equivalent but different looking answers.

§ 8.4 # 51 |

$$\int \underbrace{\ln(x^2 - x + 2)}_{u}\, \underbrace{dx}_{dv} = x\ln(x^2-x+2) - \int \frac{x(2x-1)}{x^2-x+2}\,dx$$

$$= x\ln(x^2-x+2) + \underbrace{\int \frac{x - 2x^2}{x^2-x+2}\,dx}_{*}$$

$$= x\ln(x^2-x+2) + \int\left(-2 + \frac{4-x}{x^2-x+2}\right)dx$$

$$= x\ln(x^2-x+2) - 2x - \int \frac{x-4}{x^2-x+2}\,dx$$

$$= \boxed{x\ln(x^2-x+2) - 2x - \ln\sqrt{x^2-x+2} + \frac{7}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C}$$

$$* \quad \frac{x-2x^2}{x^2-x+2} = -2 + \underbrace{\frac{4-x}{x^2-x+2}}$$

partial fractions on this part is unneeded since x^2-x+2 is irreducible.

$$\begin{array}{r} -2 \\ x^2-x+2 \,\big)\, \overline{x - 2x^2} \\ \underline{-2x^2+2x-4} \\ -x+4 \end{array}$$

$$\int \frac{x-4}{x^2-x+2}\,dx = \int \frac{x - 1/2}{(x-1/2)^2 + 3/4}\,dx - \frac{7}{2}\int \frac{dx}{x^2-x+2}$$

$$= \frac{1}{2}\ln|x^2-x+2| - \frac{7}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

using # 34.

we did this one already.

§ 8.4 # 69 Let F, G, Q be polynomials such that,

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$ for all x except when $Q(x) = 0$. $\boxed{\text{Prove: } F(x) = G(x) \text{ for all } x \in \mathbb{R}.}$

Continuity of polynomials yields

$$\lim_{x \to a} F(x) = F(a) \quad \& \quad \lim_{x \to a} G(x) = G(a)$$

for <u>any</u> $a \in \mathbb{R}$. Notice if $Q(x) \neq 0$ then we can simply multiply $\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$ to obtain $F(x) = G(x)$ for x such that $Q(x) \neq 0$. Consider then $a \in \mathbb{R}$ such that $Q(a) = 0$. Use continuity of F,

$$F(a) = \lim_{x \to a} F(x)$$

$$= \lim_{x \to a} \left(F(x) \frac{Q(x)}{Q(x)} \right) :$$ note $Q(a) = 0$ but $Q(x) \neq 0$ for $x \neq a$ if we are "close" to the limit point. (why can we get that close?)

$$= \lim_{x \to a} \big(G(x) \big) :$$ using the given fact for $x \neq a$.

$$= G(a).$$

Hence $F(a) = G(a)$ for $a \in \mathbb{R}$ s.t. $Q(a) = 0$. It follows $F = G$.

<u>Remark</u>: $$\frac{1+x}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

why do we get to equate numerators after multiplying by $x(x^2+3)$? It's like

$$\frac{F(x)}{Q(x)} = \frac{A(x^2+3) + (Bx+C)x}{Q(x)}$$ for $Q(x) = x(x^2+3)$

$F(x) = 1 + x$ and $G(x) = A(x^2+3) + (Bx+C)x$.
This problem justifies plugging in the roots!