Math 132, Test 1, September 22, 2009 Name: ______________________.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 1500 points. There is a take-home bonus problem to pick up once the test is done.

1. [150pts.] Calculate the limits below. Use proper notation and indicate when you use L'Hopital's Rule explicitly, Thanks.

(a.) $\lim_{x\to 0}\left(\frac{1-\cos(x)}{x^2}\right) \underset{(\frac{0}{0})}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{\sin(x)}{2x}\right) \underset{(\frac{0}{0})}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{\cos(x)}{2}\right) = \boxed{\frac{1}{2}.}$

(b.) $\lim_{x\to 0}(\ \sin(x)\ln(x)\ ) = \lim_{x\to 0}\left(\frac{\ln(x)}{\frac{1}{\sin x}}\right)$

$$\underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{1/x}{\frac{-\cos(x)}{\sin^2(x)}}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\sin^2(x)}{x\cos(x)}\right) \underset{(\frac{0}{0})}{\overset{L}{=}} \lim_{x\to 0}\left(\frac{-2\sin(x)\cos(x)}{\cos(x)-x\sin(x)}\right) = \boxed{0.}$$

(c.) $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{2x} = e^{\underbrace{\lim_{x\to\infty}\ln\left(1+\frac{r}{x}\right)^{2x}}_{*}} = \boxed{e^{2r}.}$

$$* = \lim_{x\to\infty}\left[\ln\left(1+\frac{r}{x}\right)^{2x}\right] = \lim_{x\to\infty}\left[2x\ln\left(1+\frac{r}{x}\right)\right]$$

$$= \lim_{x\to\infty}\left[\frac{2\ln(1+r/x)}{\frac{1}{x}}\right]$$

$$\underset{(\frac{0}{0})}{\overset{L}{=}} \lim_{x\to\infty}\left[\frac{\frac{2}{1+r/x}\left(\frac{-r}{x^2}\right)}{\frac{-1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2r}{1+r/x}\right] = \underline{2r = *}$$

2. [150pts.] Set-up the partial fractions decomposition for the following rational functions (do NOT find explicit values for A,B,C,...)

(a.) $$\frac{9}{x^3-x} = \frac{9}{x(x^2-1)} = \frac{9}{x(x+1)(x-1)} = \underline{\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}}.$$

(b.) $$\frac{9}{\underbrace{(x^2-1)}_{\substack{\text{factors}\\ \text{into}\\ (x+1)(x-1)}}\underbrace{(x^2+4)}_{\text{irred.}}} = \underline{\frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}}.$$

3. [150pts.] Integrate:

$$\int \underbrace{e^x}_{u} \underbrace{\cos(x)\,dx}_{dv} = e^x \sin(x) - \int \underbrace{e^x}_{u} \underbrace{\sin(x)\,dx}_{dv}$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

$$\Rightarrow 2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + C_1$$

$$\Rightarrow \underline{\int e^x \cos(x)\,dx = \frac{1}{2} e^x (\sin(x) + \cos(x)) + C}.$$

4. [100pts.] Integrate:

$$\int \underbrace{\ln(x)}_{u}\, \underbrace{dx}_{dv} = x \ln(x) - \int x \frac{dx}{x}$$

$$= x \ln(x) - \int dx$$

$$= \underline{x \ln(x) - x + C}.$$

5.[150pts] Determine if the following integrals converge or diverge. If the integral converges calculate the value to which it converges. Otherwise, explain why it diverges. For part (b.) sketch the integrand and verify it confirms your answer from a qualitative perspective.

a.) $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = \lim_{t\to 0^+}\left(\int_t^1 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx\right) = \lim_{t\to 0^+}\left(e^{\sqrt{1}} - e^{\sqrt{t}}\right) = \boxed{e-1}$

$$\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = \int e^u\,du = e^{\sqrt{x}} + C$$

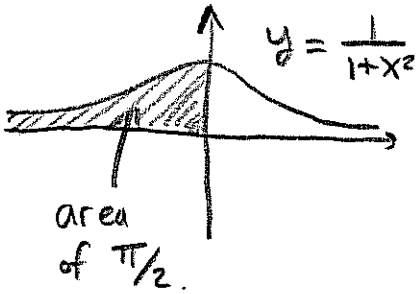
$$\boxed{\begin{array}{l} u = \sqrt{x} \\ du = \dfrac{dx}{2\sqrt{x}} \end{array}}$$

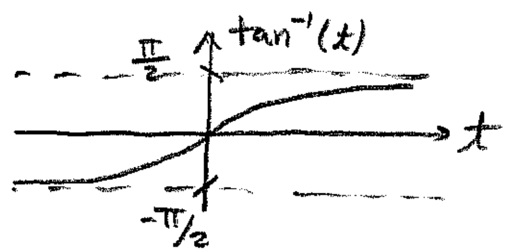
b.) $\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to -\infty}\int_t^0 \frac{dx}{1+x^2}$

$$= \lim_{t\to -\infty}\left(\tan^{-1}(0) - \tan^{-1}(t)\right)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{2}}$$





6. [200pts.] Calculate the integral below. Your answer should only have one arbitrary constant of integration.

$$\int \frac{9+x}{(x-1)(x^2+9)}\, dx$$

$$\frac{9+x}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$9+x = A(x^2+9) + (Bx+C)(x-1)$$

$$9+x = x^2(A+B) + x(-B+C) + (9A-C)$$

Equate Coefficients

$x^2\rfloor \quad 0 = A+B$ $\qquad$ $x=1 \rfloor \quad 10 = 10A \quad \therefore \quad A=1$

$x\rfloor \quad 1 = C-B$ $\qquad \Rightarrow \quad B=-1$

$x^0\rfloor \quad 9 = 9A-C$ $\qquad \Rightarrow \quad C=0$.

Thus,

$$\int \frac{9+x}{(x-1)(x^2+9)}\, dx = \int \frac{dx}{x-1} - \int \frac{x\,dx}{x^2+9}$$

$$= \ln|x-1| - \frac{1}{2}\int \frac{du}{u} \qquad u = x^2+9, \quad du = 2x\,dx$$

$$= \underline{\ln|x-1| - \frac{1}{2}\ln|x^2+9| + C}.$$

7.[150pts] Calculate

$$\int \sin^5\theta\cos^3\theta\,d\theta = \int \sin^5\theta\,(1-\sin^2\theta)\cos\theta\,d\theta$$

$$= \int u^5(1-u^2)\,du \qquad \left[\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right]$$

$$= \int (u^5 - u^7)\,du$$

$$= \frac{1}{6}u^6 - \frac{1}{8}u^8 + C$$

$$= \underline{\frac{1}{6}\sin^6\theta - \frac{1}{8}\sin^8\theta + C}.$$

8.[150pts] Calculate

$$\int_0^1 \sin^2(3\pi x)\,dx = \int_0^{3\pi} \sin^2(u)\,\frac{du}{3\pi} \qquad \begin{array}{l} u = 3\pi x \\ u(0) = 0 \\ u(1) = 3\pi \\ du = 3\pi\,dx \end{array}$$

$$= \frac{1}{6\pi}\int_0^{3\pi}\left(1-\cos(2u)\right)du$$

$$= \frac{1}{6\pi}\left[u - \frac{1}{2}\sin(2u)\right|_0^{3\pi}$$

$$= \frac{1}{6\pi}\left[3\pi - \frac{1}{2}\sin 6\pi - 0 + \frac{1}{2}\sin(0)\right]$$

$$= \frac{3\pi}{6\pi}$$

$$= \boxed{\frac{1}{2}.}$$

9.[200pts] Integrate, be sure to leave your answer as an algebraic function.

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{(27\sin^3\theta)(3\cos\theta\,d\theta)}{\sqrt{9-9\sin^2\theta}}$$

$x = 3\sin\theta$
$dx = 3\cos\theta\,d\theta$

$\underline{9-9\sin^2\theta = 9\cos^2\theta}$

$$= \int \frac{81}{3}\,\frac{\sin^3\theta\cos\theta}{\cos\theta}\,d\theta$$

$$= 27\int \sin^3\theta\,d\theta$$

$$= 27\int (1-\cos^2\theta)\sin\theta\,d\theta$$

$$= 27\int (1-u^2)(-du)$$

$$\left[\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right]$$

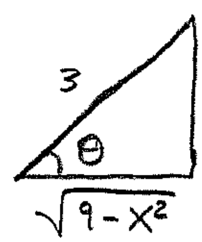
$$= 27\int (u^2-1)\,du$$

$$= 27\left[\frac{u^3}{3} - u\right] + C$$

$$= 9u^3 - 27u + C$$

$$= 9\cos^3\theta - 27\cos\theta + C$$

$$= \underline{9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - 27\left(\frac{\sqrt{9-x^2}}{3}\right) + C}$$



$\sin\theta = \frac{x}{3}$

$$= \sqrt{9-x^2}\left[9\left(\frac{9-x^2}{27}\right) - \frac{27}{3}\right] + C$$

$$= \sqrt{9-x^2}\left[\frac{1}{3}9 - \frac{1}{3}x^2 - 9\right] + C$$

$$= \sqrt{9-x^2}\left[-\frac{1}{3}x^2 - 6\right] + C = \boxed{-\frac{1}{3}\sqrt{9-x^2}\left(x^2+18\right) + C}$$

10. [50pts.] Assume $\omega \in \mathbb{R}$. Use trigonometry you remember (or derive) to state formulas for C_1 and C_2 given that the following formula implicitly defines C_1, and C_2 as functions of the amplitude A and the phase angle ϕ:

$$A\sin(\omega t + \phi) = C_1\cos(\omega t) + C_2\sin(\omega t).$$

$$\parallel$$

$$A\sin\omega t\cos\phi + A\cos\omega t\sin\phi = C_1\cos\omega t + C_2\sin\omega t$$

Comparing these equations we find

$$\boxed{\begin{aligned} C_1 &= A\sin\phi \\ C_2 &= A\cos\phi \end{aligned}}$$

11. [25pts.] Let $\omega \in \mathbb{R}$.Use trigonometry you remember (or derive) to find the value for ϕ which makes the following equation true: $\cos(\omega t) = \sin(\omega t + \phi)$.

$$\sin(\omega t + \phi) = \sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi = \cos(\omega t)$$

Need $\sin\phi = 1$ and $\cos\phi = 0$ $\Rightarrow$ $\boxed{\phi = \frac{\pi}{2}}$

(other answers possible, $\phi = \frac{\pi}{2} + 2\pi k$ for any $k \in \mathbb{Z}$ works)

12. [25pts.] Use trigonometry you remember (or derive) to find the values for A and B the following eqn. true: $2\sin(3t)\cos(4t) = \sin(At) + \sin(Bt)$.

$$2\frac{1}{2i}\left(e^{3it} - e^{-3it}\right)\frac{1}{2}\left(e^{4it} + e^{-4it}\right) =$$

$$= \frac{1}{2i}\left(e^{7it} - e^{-7it}\right) - \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$= \sin(7t) - \sin(t)$$

$$= \sin(7t) + \sin(-t)$$

$\therefore$ $\boxed{A = 7, \quad B = -1}$ or $\boxed{A = -1, \quad B = 7}$