Problem 1 [10pts] Define the items detailed below. Assume that A, B, S are sets and P, Q are propositions in what follows:

a. well ordering principle

Every <u>nonempty</u> subset of $\mathbb{N}$ has a smallest element.

b. $A \subseteq B$

$A \subseteq B$ iff for each $x \in A$ we find $x \in A \Rightarrow x \in B$.

c. contrapositive of "$P \implies Q$"

$\sim Q \Rightarrow \sim P$.

d. relation $R : A \to B$

$R \subseteq A \times B$ is a relation from A to B.

e. converse of "$P \implies Q$"

$Q \Rightarrow P$.

Problem 2 [15pts] Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Calculate:

a. $A \cup B$

$A \cup B = \{1, 2, 3, 4, 5\}$

b. $\mathcal{P}(A \cap B)$

$A \cap B = \{3\} \Rightarrow \mathcal{P}(A \cap B) = \mathcal{P}(\{3\}) = \{\emptyset, \{3\}\}$.

c. $A - B$

$A - B = \{1, 2\}$

Problem 3 [5pts] Let $x, y \in \mathbb{Z}$. Prove that if x and y are odd then $x + y$ is even.

Suppose $x, y \in \mathbb{Z}$ are odd. Then $\exists\, m, n \in \mathbb{Z}$ such that $x = 2m+1$ and $y = 2n+1$. Consider,

$$x + y = (2m+1) + (2n+1) = 2(m+n+1) = 2k$$

and $k = m+n+1 \in \mathbb{Z}$ thus $x+y$ is even.

Problem 4 [10pts] Suppose that A, B, C, D are sets.
Prove that if $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$.

Suppose $A \subseteq C$ and $B \subseteq D$. Let $(a, b) \in A \times B$ then $a \in A$ and $b \in B$. Notice $a \in A \Rightarrow a \in C$ since $A \subseteq C$. Likewise, $b \in B \Rightarrow b \in D$ since $B \subseteq D$. We find $a \in C$ and $b \in D$ thus $(a, b) \in C \times D$. Hence $A \times B \subseteq C \times D$.

Problem 5 [10pts] Prove that $3 | (n^3 + 5n + 6)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be the proposition that 3 divides $n^3 + 5n + 6$.

Observe that $P(1)$ is true since $1+5+6 = 12 = 3(4)$.

Assume $P(n)$ is true. (this is the inductive hypothesis)

By induction hypothesis $\exists k \in \mathbb{Z}$ s.t. $n^3 + 5n + 6 = 3k$. Consider,

$$\begin{aligned}
(n+1)^3 + 5(n+1) + 6 &= n^3 + 3n^2 + 3n + 1 + 5n + 5 + 6 \\
&= (n^2 + 5n + 6) + 3n^2 + 3n + 6 \\
&= 3k + 3n^2 + 3n + 3 \cdot 2 \\
&= 3(\underbrace{k + n^2 + n + 2}_{\text{call this } l.})
\end{aligned}$$

clearly $l \in \mathbb{Z}$.

Thus $3 | (n+1)^3 + 5(n+1) + 6$

We find $P(n) \Rightarrow P(n+1)$ for arbitrary n hence by PMI $P(n)$ true $\forall n \in \mathbb{N}$.

Problem 6 [15pts] Let $S = \{-2, -1, 0, 1, 2\}$. Define a relation R on S as follows:

$$R = \{(x, y) \mid x, y \in S \text{ with } x^2 = y^2\}.$$

Prove that R is an equivalence relation on S. Then find the partition of S that corresponds to this equivalence relation.

Clearly $R \subseteq S \times S$ thus R is a relation.
Moreover,

(1) $x^2 = x^2 \ \forall x \in S$ thus $xRx \ \forall x \in S$
Thus R is reflexive

(2) Let $x, y \in S$ and suppose xRy then
$x^2 = y^2 \Rightarrow y^2 = x^2 \Rightarrow yRx \ \therefore$ R symmetric

(3) Let $x, y, z \in S$ such that xRy and yRz then
$x^2 = y^2$ and $y^2 = z^2 \Rightarrow x^2 = z^2 \Rightarrow xRz$.
$\therefore$ R transitive.

Thus R is an equivalence relation.
The partition follows from the equivalence classes as we discussed in lecture,

$$\overline{2} = \{x \in S \mid xR2\} = \{x \in S \mid x^2 = 2^2\}$$
$$\overline{1} = \{x \in S \mid x^2 = 1^2\}$$
$$\overline{0} = \{x \in S \mid x^2 = 0\}$$

Thus $\overline{2} = \{-2, 2\}$ and $\overline{1} = \{-1, 1\}$, $\overline{0} = \{0\}$.
Clearly $S = \overline{0} \cup \overline{1} \cup \overline{2}$ and $\underbrace{\mathcal{R} = \{\{0\}, \{-1, 1\}, \{-2, 2\}\}}_{\text{the partition}}$

Problem 7 [5pts] Give a *useful* denial of "roses are not red and violets are blue"

"roses are red or violets are not blue"

Problem 8 [10pts] Given propositions P and Q, **prove**

$$\boxed{\sim (P \vee Q) \Leftrightarrow \sim P \wedge \sim Q.}$$

Justify your logic with a truth table.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$\sim P \wedge \sim Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

thus $\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ are logically equivalent.

Problem 9 [5pts] **Prove or disprove the following statement:**

$$\exists x \in \mathbb{R} \text{ such that } x^2 - 1 \leq 0.$$

Notice $1^2 - 1 = 0$ $\therefore$ $\exists x \in \mathbb{R}$ such that $x^2 - 1 \leq 0$.

Problem 10 [5pts] Let A be a set and suppose that R is an equivalence relation on A. **Prove or disprove:** R^{-1} is an equivalence relation on A.

Sol^{n} (I) Assume $R \subseteq A \times A$ is an equivalence relation on A. Recall that we define

$$R^{-1} = \{(x,y) \mid (y,x) \in R\}.$$

Note $(a,b) \in R^{-1}$ iff $(b,a) \in R$ iff $(a,b) \in R$.
Therefore R and R^{-1} have the same points; $R = R^{-1}$.
Hence R^{-1} is an equivalence relation.

Sol^{n} (II) $R^{-1} \subseteq A \times A$ by def^{n} of R^{-1}.

① Notice $xRx \Rightarrow xR^{-1}x$ $\therefore$ R^{-1} reflexive.

② Suppose $xR^{-1}y \Rightarrow yRx \Rightarrow xRy \Rightarrow yR^{-1}x$.
for all $x, y \in A$ $\therefore$ R^{-1} symmetric.

③ Suppose $xR^{-1}y$ and $yR^{-1}z$ then by def^{n} of R^{-1},
yRx and $zRy \Rightarrow zRx$ by transitivity of R
Hence $xR^{-1}z$ $\therefore$ R^{-1} transitive

Consequently we can conclude R^{-1} is an equivalence relation.

(Sol^{n} (I) is due to my brother Bill.)

do not consult others for help, you may use your notes and the text but that is all. (Due Wednesday by 4pm, slide under my door or give it to me directly, thanks!)

Problem 11 [10pts] Prove by induction on n that

$$\left.\frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) = \frac{(-1)^n n!}{(x-1)^{n+1}}\right\} \text{ call this } P(n)$$

for all $n \in \mathbb{N}$.

The proposition is clearly true for $n=1$ since

$$\frac{d}{dx}\left(\frac{1}{x-1}\right) = \frac{-1}{(x-1)^2} = \frac{(-1)^1\, 1!}{(x-1)^{1+1}}.$$

Assume that $\frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) = \frac{(-1)^n n!}{(x-1)^{n+1}}$ for some $n \geq 1$.

Consider,

$$\frac{d^{n+1}}{dx^{n+1}}\left(\frac{1}{x-1}\right) = \frac{d}{dx}\left(\frac{d^n}{dx^n}\left(\frac{1}{x-1}\right)\right)$$

$$= \frac{d}{dx}\left(\frac{(-1)^n\, n!}{(x-1)^{n+1}}\right) \quad : \text{ using the induction hypothesis.}$$

$$= (-1)^n n! \frac{d}{dx}\left((x-1)^{-(n+1)}\right)$$

$$= (-1)^n\, n!\,(-(n+1))\,(x-1)^{-(n+1)-1}$$

$$= \frac{(-1)^{n+1}(n+1)!}{(x-1)^{(n+1)+1}} \Rightarrow P(n) \Rightarrow P(n+1)$$

$\therefore$ $P(n)$ true $\forall n \in \mathbb{N}$ by PMI.