Math 221	Mission 2: matrix algebra & theory of linear equations (40+5)pts)

Same instructions as Mission 1. This homework is based on Lectures 4 - 9. There are 5pts to earn for completely following formatting instructions.

Problem 21: (1pt) The commutator of a square matrices A, B is given by $[A, B] = AB - BA$. Show $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ holds for arbtrary matrices A, B, C (show work below):

Problem 22: The trace of a square matrix is defined by $tr(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in \mathbb{R}^{n \times n}$ (show work below:)

(a.) (1pt) prove $tr(cA + B) = ctr(A) + tr(B)$

(b.) (2pt) Let $X \in \mathbb{R}^{m \times p}$ and $Y \in \mathbb{R}^{p \times m}$. Prove $tr(XY) = tr(YX)$

(c.) (1pt) find all A, B for which $[A, B] = I$

Problem 23: (5pt) Consider the matrices below:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

Calculate if possible, or explain why the calculation is undefined: (show work below)

(a.) $(4B)C + 2B$

(b.) $(-AC)^T + 5D^T$

(c.) $(BA^T - 2C)^T$

(d.) $B^T(CC^T - A^TA)$

(e.) $D^TE^T - (ED)^T$

Problem 24: (3pts) Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 0 & 1 \end{bmatrix}$ let e_1, e_2, e_3, e_4 denote the standard basis for $\mathbb{R}^4$ and $\bar{e}_1, \bar{e}_2, \bar{e}_3$ denote the standard basis for $\mathbb{R}^3$. Calculate the following if possible, or explain why the proposed calculation is undefined: (show work below if you can do it neatly)

(a.) $A[e_1|e_4]$

(b.) $tr[Ae_2|Ae_3|Ae_4]$

(c.) $\bar{e}_1 A e_3$

(d.) $A^T \bar{e}_1$

(e.) $tr\left([(\bar{e}_1^T A)^T | (\bar{e}_2^T A)^T] \left[\frac{\bar{e}_2^T A}{\bar{e}_3^T A} \right] \right)$

(f.) $A + 6\bar{e}_1 e_1^T + 6\bar{e}_1 e_2^T - 2\bar{e}_1 e_3^T - 3\bar{e}_1 e_4^T - 4\bar{e}_1 e_2^T - 5\bar{e}_2 e_2^T - 2\bar{e}_3 e_1^T - 2\bar{e}_3 e_2^T - \bar{e}_3 e_4^T$

Problem 25: (1pts) A 3×3 antisymmetric matrix A has $e_2^T A e_3 = 1$ and $col_1(A) = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$. Find A explicitly.

Problem 26: (1pt) Let $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$. Prove $(cA+B)^T = cA^T + B^T$ (show work below):

Problem 27: (1pts) Solve $3x+5y = a$ and $4x+7y = b$ for x and y in terms of arbitrary a, b. If you use row-reduction here, I kill you. (show work below):

Problem 28: (1pts) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $A^{-1}B^T = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ then find B. (show work below)

Problem 29: (2pts) Let A be as in the problem above and define $M = \begin{bmatrix} A^T & 0 \\ 0 & A^{-1} \end{bmatrix}$. If $N = M^{-1}$ has block-decomposition $N = \begin{bmatrix} C & D \\ E & F \end{bmatrix}$ then find C, D, E, F.

Problem 30: (1pts) For what choice of C do the lines $x - y = 2$ and $x + y = 4$ and $x + 2y = C$ have a common point of intersection ?

Problem 31: (1pt) Let $rref[v_1|v_2|v_3|e_3|e_2 + e_3|e_1] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 7 \\ 0 & 1 & 0 & 0 & 2 & 6 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right]$. If $A = [v_1|v_2|v_3]$ then calculate A^{-1} in terms of v_1, v_2, v_3.

Problem 32: (1pt) Let $a, b, c \neq 0$ and suppose $M = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix}$. Calculate M^{-1}.

Problem 33: (2pt) If possible, write M as the product of elementary matrices. If not, explain why it is not possible.

(a.) $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(b.) $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Problem 34: (1pt) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Calculate A^{-1} and be sure to simplify your answer.

Problem 35: (2pt) Let $a, b, c \in \mathbb{R}$. Solve $\left\{ \begin{array}{rcl} x_1 + x_2 + x_3 & = & a \\ 2x_1 + 2x_3 & = & b \\ 3x_2 + 3x_3 & = & c \end{array} \right\}$ and calculate $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}^{-1}$

Problem 36: (1pts) If $\text{rref}[\ v_1 \mid v_2 \mid v_3 \mid b_1\ ] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ where $[v_1|v_2|v_3] = \begin{bmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \\ 9 & 9 & 9 \end{bmatrix}$ then find b_1.

Problem 37: (2pts) You are given that:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 3 & 6 & 1 & 4 & 2 \\ -2 & -4 & 1 & -1 & -3 \end{bmatrix} \quad \text{has} \quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a.) Find the basis for $\text{Col}(A)$ using columns of A via the CCP,

(b.) Find the basis for $\text{Null}(A)$.

Problem 38: (2pt) Suppose $A = [v_1|v_2|v_3|v_4] \in \mathbb{R}^{4\times 4}$ where $v_2 - v_1 = 6v_3 + 7v_4$. Find infinitely many solutions of $Ax = 0$. What condition is needed to be sure you have all the solutions possible ?

Problem 39: (2pt) Show $A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is nilpotent and show $(I - A)^{-1} = c_1 I + c_2 A + c_3 A^2$ for appropriate choices of the constants c_1, c_2, c_3.

Problem 40: (1pt) Let $\sim$ denote row-equivalence; $A \sim B$ if there is a finite sequence of elementary row operations which changes A to B. Consider $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 12 & 1 & 1 \\ 7 & 14 & 8 & 0 \\ 10 & 20 & 7 & 1 \end{bmatrix}$.
Is $A \sim B$? If so, find k-elementary matrices for which $B = E_1 E_2 \cdots E_k A$