Same instructions as Mission 1. This homework is based on Lectures 10 - 15. There are 5pts to earn for completely following formatting instructions. Feel free to use technology for any row-reductions, however, realize you may need to do some of these calculations in your next Boss Fight.

Problem 41: (5pts) For each map defined below,find the standard matrix and determine if the map is injective, surjective or neither.

(a.) $S(x, y, z) = (x + 2y, y + 3z)$

(b.) $T(x_1, x_2, x_3, x_4) = (\ x_1 + x_4,\ x_2 + x_3,\ x_4,\ x_1 - x_4, x_1 - x_4)$

(c.) $R(x, y) = (y, x)$

(d.) $R \circ S$

(e.) S^2

Problem 42: (1pt) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation for which $T(1,1,1) = (2,3,4)$ and $T(0,1,0) = (3,3,3)$ and $T(0,1,1) = (0,0,7)$. Find the standard matrix of T.

Problem 43: (5pt) In each case below we study $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(v) = Av$. For each of the following, plot the unit-square and the image of the square under the transformation. Please use color-coding to illustrate which edge goes to which edge.

$$(\mathbf{a.})\ A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\quad (\mathbf{b.})\ A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix},\quad (\mathbf{c.})\ A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix},\quad (\mathbf{d.})\ A = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix},\quad (\mathbf{e.})\ A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Problem 44: (1pt) We define $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. Calculate $\det(A)$ for each case in the previous problem and determine what this has to do with the area of the transformed region. (you can solve Problems 43 and 44 together, no need to put answers here)

Problem 45: (4pt) Show W is a subspace using a theorem from lecture. (show work below)

(a.) $W = \{(s, s+t, s-t) \mid s,t \in \mathbb{R}\} \leq \mathbb{R}^3$

(b.) $W = \{(x,y,z,w) \in \mathbb{R}^4 \mid x+y=0, z+w=0\} \leq \mathbb{R}^4$

(c.) $W = \{a + bx^2 \mid a,b \in \mathbb{R}\} \leq \mathbb{R}[x]$

(d.) $W = \{A \in \mathbb{R}^{n\times n} \mid tr(A) = 0\} \leq \mathbb{R}^{n\times n}$

Problem 46: (2pt) Determine if the set S below is linearly independent, or linearly dependent. Give evidence for your claims. (show work below)

(a.) $S = \{1 + x,\ x^2 + x,\ x^2 - 1\}$

(b.) $S = \{1 - x,\ x - x^2,\ x^2 - x^3\}$

Problem 47: (2pt) Every vector space has a basis and we define the number of elements in the basis to be the **dimension** of the vector space. If there is no finite basis then the vector space is said to be **infinite dimensional**. For example, $\mathbb{R}[x]$ is infinite dimensional whereas $P_2(\mathbb{R}) = span\{1, x, x^2\}$ is three dimensional.

(a.) What is the dimension of $W = \{f(x) \in \mathbb{R}[x] \mid f''(x) = 0\}$?

(b.) If $rank(A) = 3$ for $A \in \mathbb{R}^{6\times 7}$ then what is the dimension of $Null(A)$?

Problem 48: (2pt) If W is a k-dimensional subspace of $\mathbb{R}^n$ then there are two ways to describe W in terms of matrix subspaces. We can either find $A \in \mathbb{R}^{m\times n}$ for which $W = Col(A)$ or we can find $B \in \mathbb{R}^{n\times p}$ for which $W = Null(B)$. For the subspaces below, find such A and B (answers not unique, grader has to earn their money here)

(a.) $W = \{(t, t, t, t) \mid t \in \mathbb{R}\}$

(b.) $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y = z + w\}$

Problem 49: (2pt) A **line** through the origin is a one-dimensional subspace of $\mathbb{R}^n$ and a **plane** containing the origin is a two-dimensional subspace of $\mathbb{R}^n$. For the contexts given below, determine the least number of equations which are needed to specify the given object:

(a.) A line in $\mathbb{R}^3$

(b.) A plane in $\mathbb{R}^3$

(c.) A line in $\mathbb{R}^4$

(d.) A plane in $\mathbb{R}^4$

Problem 50: (2pt) Given $T(v) = Av$ for A below, determine if T is invertible. If not, explain why not. If so, find the formula for $T^{-1}(a, b, c)$

(a.) $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$

(b.) $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix}$

Problem 51: (1pt) Let $\beta = \{(1, 4, -1), (2, 7, 1), (1, 3, 0)\}$ serve as the basis for $\mathbb{R}^3$. If

$$(x, y, z) = \bar{x}(1, 4, -1) + \bar{y}(2, 7, -1) + \bar{z}(1, 3, 0)$$

then find formulas for $\bar{x}, \bar{y}, \bar{z}$ in terms of x, y, z. In other words, calculate $\Phi_\beta(x, y, z)$.

Problem 52: (2pt) Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $T(f(x)) = xf(x) + f'(x)$. Define $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$ and calculate $[T]_{\beta,\gamma}$.

Problem 53: (2pt) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x+y+z, y+z)$. Define $\beta = \{(1, 1, 1), (1, 0, 0), (0, 1, -1)\}$ and $\gamma = \{(0, 1), (1, 0)\}$ and calculate $[T]_{\beta,\gamma}$.

Problem 54: (2pt) Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $T(A) = A^T$. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\gamma = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$. Calculate $[T]_{\beta,\beta}$ and $[T]_{\gamma,\gamma}$.

Problem 55: (1pt) Let $R_z(\theta)(x, y, z) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta, z)$ where $\theta \in \mathbb{R}$. This defines a rotation about the z-axis by angle θ in the counter-clockwise sense (CCW). Find the formula for a rotation $R_y(\beta)$ about the y-axis by angle β in the CCW-sense and find $[R_z(\theta)]$ and $[R_y(\beta)]$.

Problem 56: (1pt) Given θ, β are not integer multiples of π, show that $R_y(\beta) \circ R_z(\theta) \neq R_z(\theta) \circ R_y(\beta)$. (show work below)

Problem 57: (1pt) Find a matrix P for which $R_z(\theta) = P^{-1}[R_y(\beta)]P$.

Problem 58: (1pt) It can be shown that any matrix R for which $R^T R = I$ and $det(R) = 1$ is a matrix of a rotation by some angle θ. In other words, if $T(v) = Rv$ then there exists a basis γ for which $[T]_{\gamma,\gamma} = [R_z(\theta)]$. By what θ does R rotate ?

Problem 59: (1pt) Let $\beta = \{v_1, v_2\}$ be a basis for $\mathbb{R}^2$ such that $T(v_1) = 6v_1$ and $T(v_2) = v_1 + 7v_2$ for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. If $[T] = \begin{bmatrix} 7 & 0 \\ 2 & 6 \end{bmatrix}$ then, if possible, find β (possibly non-unique)

Problem 60: (2pt) A **line** with **base-point** p and **direction-vector** $v \neq 0$ is denoted $\mathcal{L}_p(v) = \{p + tv \mid t \in \mathbb{R}\}$. Likewise, a **line-segment** from p to q is defined by $\overline{pq} = \{p + t(q - p) \mid 0 \leq t \leq 1\}$. The answers to the questions posed below may require conditions for the answer to be yes. Determine the necessary conditions to answer in the affirmative.

(a.) Let $T(x, y) = (x - y, x + y, x)$. Is $T(\overline{pq})$ a line-segment and is $T(\mathcal{L}_p(v))$ a line ?

(b.) Let $T(x, y, z) = (x + y, y + z)$. Is $T(\overline{pq})$ a line-segment and is $T(\mathcal{L}_p(v))$ a line ?