I provide space for you to work out the problems. If I happen to provide too little for a given problem, simply record your answer near the problem statement and attach your work on the next page. Staple all the pages in order together when you are finished. [2pts per problem]

Problem 1 For each system below, find a matrix A and a vector of variables v such that the systems below are equivalent to $Av = b$.

a.) $x_1 + x_2 = 1,\quad x_2 - x_3 = 2,\quad x_3 + x_4 = 3,\quad x_4 - x_5 = 4.$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{b}$$

b.) $BX + XB^TB = I_2$ where $X = \begin{bmatrix} t & x \\ y & z \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 4 & 7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} t & x \\ y & z \end{bmatrix} + \begin{bmatrix} t & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} t & x \\ 4t + 7y & 4x + 7z \end{bmatrix} + \begin{bmatrix} t & x \\ y & z \end{bmatrix}\begin{bmatrix} 5 & 28 \\ 28 & 49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} t & x \\ 4t + 7y & 4x + 7z \end{bmatrix} + \begin{bmatrix} 5t + 28x & 28t + 49x \\ 5y + 28z & 28y + 49z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$6t + 28x = 1,\quad 28t + 50x = 0,\quad 4t + 12y + 28z = 0,\quad 4x + 28y + 56z = 1$

$$A \rightarrow \begin{bmatrix} 6 & 28 & 0 & 0 \\ 28 & 50 & 0 & 0 \\ 4 & 0 & 12 & 28 \\ 0 & 4 & 28 & 56 \end{bmatrix}\underbrace{\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}}_{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow b$$

Problem 2 Find the cubic polynomial(s) for which the graph passes through $(0,1)$, $(1,3)$ and there is a horizontal tangent through $(4,-2)$.

$f(x) = Ax^3 + Bx^2 + Cx + D \rightarrow \frac{df}{dx} = 3Ax^2 + 2Bx + C$

$f(0) = 1 = D$

$f(1) = 3 = A + B + C + D$

$f(4) = -2 = 64A + 16B + 4C + D$

$f'(4) = 0 = 48A + 8B + C$

$$\text{rref}\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 64 & 16 & 4 & 1 & -2 \\ 48 & 8 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 53/144 \\ 0 & 1 & 0 & 0 & -397/144 \\ 0 & 0 & 1 & 0 & 79/18 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \therefore \boxed{f(x) = \frac{53}{144}x^3 - \frac{397}{144}x^2 + \frac{79}{18}x + 1}$$

Problem 3 Let $f(x,y,z) = \sqrt[3]{x^2+y^2+z^2}$. Calculate the linearization of f based at $(10,0,5)$. In other words, work out the details of:

$$L_f^p(x,y,z) = f(p) + f_x(p)(x-10) + f_y(p)(y-0) + f_z(p)(z-5)$$

where $f_x = \frac{\partial f}{\partial x}$ etc. Use your result to approximate the cube root of ~~126~~. Notice, $\sqrt[3]{125} = 5$. Hint: $y = 1$.
136 more fun.

$$\frac{\partial f}{\partial x} = \frac{1}{3}(x^2+y^2+z^2)^{-2/3}(2x) \therefore f_x(10,0,5) = \frac{20}{3}(125)^{-2/3} = \frac{20}{75} = \frac{4}{15}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{3}(x^2+y^2+z^2)^{-2/3}(2y) \therefore f_y(10,0,5) = 0.$$

$$\frac{\partial f}{\partial z} = \frac{1}{3}(x^2+y^2+z^2)^{-2/3}(2z) \therefore f_z(10,0,5) = \frac{10}{3}(125)^{-2/3} = \frac{10}{75} = \frac{2}{15}$$

Notice, $f(10,0,5) = \sqrt[3]{100+0+25} = \sqrt[3]{125} = 5$,

$$\boxed{L_f^{(10,0,5)}(x,y,z) = 5 + \frac{4}{15}(x-10) + \frac{2}{15}(z-5)}$$

$$136 = 10^2 + 0^2 + 6^2$$

$$L_f^{(10,0,5)}(10,0,6) = 5 + \frac{4}{15}(10-10) + \frac{2}{15}(6-5)$$

$$\therefore \underline{\sqrt[3]{136} \approx 5 + \frac{2}{15} = \frac{77}{15}}.$$

$$\sqrt[3]{136} \cong 5.143 \quad \text{vs.} \quad \frac{77}{15} = 5.1333\ldots$$

Problem 4 Let $F(x,y,z) = (x^2+y^2, y+z^2, yz)$ and calculate:

(a.) the Jacobian matrix of F; $J_F = [\partial_x F | \partial_y F | \partial_z F]$

(b.) the linearization of F at the point (a,b,c)

(c.) the inverse function theorem of advanced calculus says a local inverse of F exists only if J_F is invertible at (a,b,c). Where can we be sure F has a local inverse? (Hint: determinants are nice)

(a.) $$J_F = \begin{bmatrix} 2x & 2y & 0 \\ 0 & 1 & 2z \\ 0 & z & y \end{bmatrix}$$

(b.) $$L_f^{(a,b,c)} = F(a,b,c) + J_F(a,b,c)\begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$$

$$= (a^2+b^2, b+c^2, bc) + \begin{bmatrix} 2a & 2b & 0 \\ 0 & 1 & 2c \\ 0 & c & b \end{bmatrix}\begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$$

$$= \boxed{\begin{aligned} (&a^2+b^2+2a(x-a)+2b(y-b), \\ &b+c^2+y-b+2c(z-c), \\ &bc+c(y-b)+b(z-c)) \end{aligned}}$$

(c.) $J_F(a,b,c)^{-1}$ exists $\Leftrightarrow \det(J_F(a,b,c)) \neq 0$

$$\det\begin{bmatrix} 2a & 2b & 0 \\ 0 & 1 & 2c \\ 0 & c & b \end{bmatrix} = 2a\det\begin{bmatrix} 1 & 2c \\ c & b \end{bmatrix} = \boxed{2a(b-2c^2) \neq 0}$$

any (a,b,c) for which this condition is met the inverse fnct. Th^m gives local inverse exists.

Problem 5 Let $ax + by = f$ and $cx + dy = g$. Assume this system has unique solution. Solve it via:

(a.) Cramer's Rule

(b.) multiplication by the inverse of the coefficient matrix

(a.) $\begin{aligned} ax + by &= f \\ cx + dy &= g \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$

$$x = \frac{\det \begin{bmatrix} f & b \\ g & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \boxed{\frac{fd - bg}{ad - bc}}$$

$$y = \frac{\det \begin{bmatrix} a & f \\ c & g \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \boxed{\frac{ag - cf}{ad - bc}}$$

(b.) Then we can also solve by multiplying by $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A\vec{v} = \vec{b} \Rightarrow A^{-1}A\vec{v} = A^{-1}\vec{b}$$

$$\therefore \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix}$$

$$\Rightarrow \boxed{x = \frac{df - bg}{ad-bc} \ \& \ y = \frac{ag - cf}{ad-bc}}$$

Problem 6 Use row reduction to find the $rref(A)$ for $A = \begin{bmatrix} 0 & 0 & 2 & 1 & -1 & 0 \\ 2 & 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 3 & 1 & 0 \end{bmatrix}$

$$A \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 2 & 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 2 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 2 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 2 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_2 + r_3]{r_1 + 2r_3} \begin{bmatrix} 2 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow[-r_3]{r_1/2,\ r_2/2} \boxed{\begin{bmatrix} 1 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = rref(A)}$$

Problem 7 Find the solution set of the following system of linear equations:

$$\begin{aligned} 2x_3 + x_4 - x_5 &= 0, \\ 2x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 0, \\ 2x_1 + x_2 + 2x_3 + 3x_4 + x_5 &= 0. \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 1 & -1 & 0 \\ 2 & 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 3 & 1 & 0 \end{array}\right] \sim \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad \underline{\text{by P18}}$$

Hence, $\boxed{\text{Sol}^{n} \text{ Set} = \left\{ \left(-\tfrac{1}{2}x_2 - x_5, x_2, \tfrac{1}{2}x_5, 0, x_5\right) \middle| x_2, x_5 \in \mathbb{R} \right\}}$

Problem 8 Find the solution sets of systems **I.** and **II.** given below:

$$\textbf{I.} \left\{ \begin{aligned} 2x_3 + x_4 &= -1, \\ 2x_1 + x_2 + 2x_3 + 2x_4 &= 1, \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= 1. \end{aligned} \right\} \quad \& \quad \textbf{II.} \left\{ \begin{aligned} 2x_3 + x_4 &= 0, \\ 2x_1 + x_2 + 2x_3 + 2x_4 &= 0, \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= 0. \end{aligned} \right\}$$

$$\left[\begin{array}{cccc|cc} 0 & 0 & 2 & 1 & -1 & 0 \\ 2 & 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|cc} 1 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\text{(I.)} \quad \begin{aligned} x_1 &= 1 - \tfrac{1}{2}x_2 \\ x_2 &= x_2 \\ x_3 &= -\tfrac{1}{2} \\ x_4 &= 0 \end{aligned} \qquad \text{(II.)} \quad \begin{aligned} x_1 &= \tfrac{-1}{2}x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$\boxed{\text{Sol}^{n} \text{ Set } I = \left\{ \left(1 - x_2/2, x_2, -1/2, 0\right) \middle| x_2 \in \mathbb{F} \right\}}$ // $\boxed{\text{Sol}^{n}_{II} = \left\{ \left(\tfrac{-1}{2}x_2, x_2, 0, 0\right) \middle| x_2 \in \mathbb{F} \right\}}$

Problem 9 Consider the matrix $M = \begin{bmatrix} a & 3 & 0 \\ 3 & a & 0 \\ 0 & 0 & b \end{bmatrix}$. Find what condition is needed on the constants a, b in order that M be invertible. Given that condition, find the formula for M^{-1}.

$$\det(M) = \det\begin{bmatrix} a & 3 \\ 3 & a \end{bmatrix}\det(b) = (a^2-9)b = (a-3)(a+3)b$$

Hence we need $a \neq \pm 3$ and $b \neq 0$ to give M^{-1} exists.

Supposing $a \neq \pm 3$ and $b \neq 0$,

$$M^{-1} = \left[\begin{array}{c|c} \begin{bmatrix} a & 3 \\ 3 & a \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & b^{-1} \end{array}\right] = \left[\begin{array}{c|c} \frac{1}{a^2-9}\begin{bmatrix} a & -3 \\ -3 & a \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1/b \end{array}\right]$$

aka.

$$\boxed{M^{-1} = \frac{1}{b(a^2-9)}\left[\begin{array}{cc|c} ab & -3b & 0 \\ -3b & ab & 0 \\ \hline 0 & 0 & a^2-9 \end{array}\right]}$$

Problem 10 Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find elementary matices $E_1, E_2, \ldots, E_k$ for which $A = E_1 E_2 \cdots E_k$.

The way to do this is to keep track of the elementary matrices needed to row-reduce A to the identity matrix. Check your answer via matrix multiplication to be sure you're right.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{r_3/3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{A} = I$$

$$A = E_1^{-1} E_2^{-1} I = \boxed{\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{E_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \stackrel{\checkmark}{=} A}$$

Problem 11 Let $Q(x, y, z) = x^2 + y^2 + 2xz + 2yz - z^2$. If $v = (x, y, z)$ is a column vector as usual and $Q(v) = v^T A v$ where $A^T = A$ **find** A. Here A is a 3×3 matrix. I'll get you started, $A_{11} = 1$ and $A_{13} = 1$.

$$Q(x,y,z) = [x,y,z]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x,y,z]\begin{bmatrix} x+z \\ y+z \\ x+y-z \end{bmatrix}$$

$$= x(x+z) + y(y+z) + z(x+y-z)$$

$$= \underline{x^2 + 2xz + y^2 + 2yz - z^2}. \checkmark$$

$$\therefore \boxed{A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}$$

Problem 12 Suppose A has rank 2 and B has rank 3. Determine what you can say about:

(a) rank(AB)

(b) rank$(3A)$

(c) rank$(A+I)$

(a.) Thm in Shores, rank $(AB) \leq \min(\text{rank} A, \text{rank} B)$

Hence rank $(AB) \leq \min(2, 3) = 2$

$\therefore$ $\boxed{\text{rank}(AB) \leq 2}$

(b.) rank $(3A)$ = rank (A) (multiplying all entries by 3 does not change linear dep. between columns.)

$\therefore$ $\boxed{\text{rank}(3A) = 2}$

(c.) rank $(A+I)$ could be many things. Take A 3×3 for examples, the following A have rank 2,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$A+I = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rank } 1} \qquad A+I = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rank } 3} \qquad A+I = \underbrace{\begin{bmatrix} 0 & 3 & 0 \\ 0 & 5 & 0 \\ 0 & 5 & 1 \end{bmatrix}}_{\text{rank } 2}.$$

Problem 13 This problem borrowed from Anton's Elementary Linear Algebra text:

In Exercises **20–27**, evaluate the determinant, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

20. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$ **21.** $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ **22.** $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$

23. $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$ **24.** $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$

20.) $\det\begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} = -\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -(-6) = \boxed{6.}$

21.) $\det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boxed{-6}$

two swaps.

22.) $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{bmatrix} = 0$ rows 1 and 3 linearly dependent.

23.) $\det\begin{bmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{bmatrix} = -12\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-12)(-6) = \boxed{72}$

24.) $\det\begin{bmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{bmatrix} \xrightarrow{r_1+r_2} = \det\begin{bmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{bmatrix} = -\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boxed{6}$

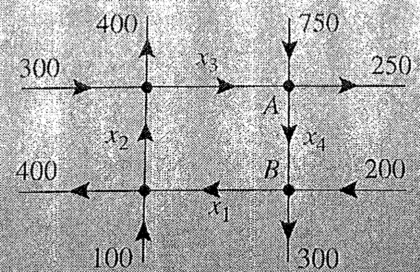
Problem 14 This problem borrowed from Anton's Elementary Linear Algebra text:

3. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.

(a) Set up a linear system whose solution provides the unknown flow rates.

(b) Solve the system for the unknown flow rates.

(c) If the flow along the road from A to B must be reduced for construction, what is the minimum flow that is required to keep traffic flowing on all roads?



◀ Figure Ex-3

what goes in also goes out:

(a.)
$$300 + x_2 = 400 + x_3$$
$$x_3 + 750 = 250 + x_4$$
$$100 + x_1 = x_2 + 400$$
$$x_4 + 200 = x_1 + 300$$

$$\boxed{\begin{aligned} x_2 - x_3 &= 100 \\ x_3 - x_4 &= -500 \\ x_1 - x_2 &= 300 \\ x_1 - x_4 &= -100 \end{aligned}}$$

(b.)
$$\text{rref}\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & 0 & -1 & -100 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= x_4 - 100 \\ x_2 &= x_4 - 400 \\ x_3 &= x_4 - 500 \end{aligned}}$$

(for $x_4 \geq 500$ since negative flow on one-way street is bad news!)

(c.) $x_4 = 500$

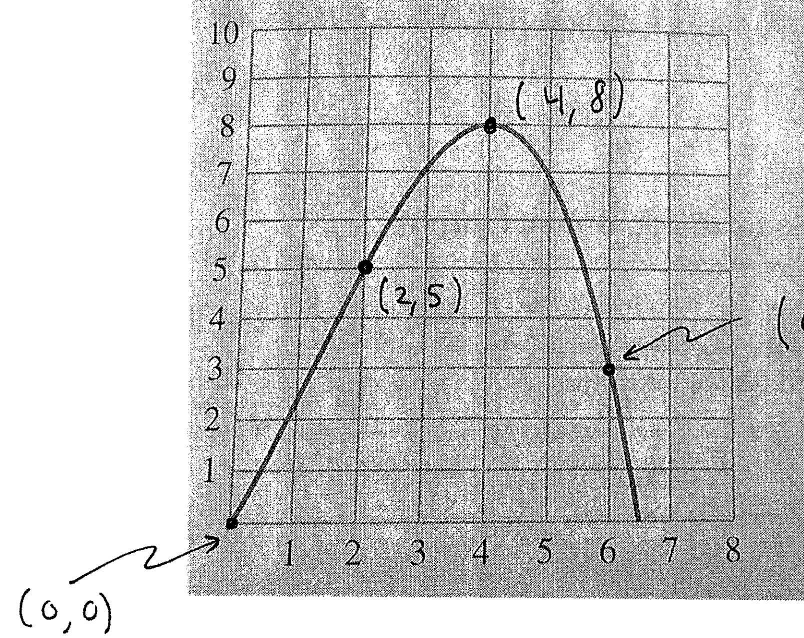
or

$x_4 = 501$

(accept either, or if some makes a different case let me know if reasonable ...)

Problem 15 This problem borrowed from Anton's Elementary Linear Algebra text:

. The accompanying figure shows the graph of a cubic polynomial. Find the polynomial.



◀ Figure Ex-16

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f(0) = \underline{d = 0}.$$

$$f(2) = 8a + 4b + 2c + d = 5 \longrightarrow 8a + 4b + 2c = 5$$

$$f(4) = 64a + 16b + 4c + d = 8 \longrightarrow 64a + 16b + 4c = 8$$

$$f(6) = 216a + 36b + 6c + d = 3 \longrightarrow 216a + 36b + 6c = 3$$

$$\text{rref}\left[\begin{array}{ccc|c} 8 & 4 & 2 & 5 \\ 64 & 16 & 4 & 8 \\ 216 & 36 & 6 & 3 \end{array}\right] = \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/8 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array} \begin{array}{l} \leftarrow a \\ \leftarrow b \\ \leftarrow c \end{array}$$

Thus, $\boxed{f(x) = -\frac{1}{8}x^3 + \frac{1}{2}x^2 + 2x}$