Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

These problems cover the material from Chapter 3 and 4 of Shores' text. However, it is more important that you study my Lecture Notes on this material. In particular, Chapters 4,5 and §7.1 – 7.7 and §7.12. Each problem is worth 3pts. You can earn 99pts on this assignment.

✓**Problem 1** Consider $W = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$. Is W a subspace of $\mathbb{R}^{2\times 2}$? If the answer is yes then find a basis for W and state the dimension of W.

✓**Problem 2** Suppose $W = span\{v_1, v_2, v_3\} \subseteq \mathbb{R}^n$ for $n > 3$. Is W a subspace? If the answer is yes then state the possible dimensions of W.

✓**Problem 3** Let $\beta = \{1, x^2, x\}$ be the **ordered** basis for P_2. Find the coordinate vector of $f(x) = (x-3)^2$.

✓**Problem 4** For A given below:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Find,

(a.) a basis for $Col(A)$

(b.) a basis for $Null(A)$; (basis for the solution set of $Ax = 0$)

(c.) solve $Ax = (1,1,1,1)$ if possible.

(d.) extend the basis of $Col(A)$ to obtain a basis for $\mathbb{R}^4$.

✓**Problem 5** Consider P_2 with basis $\{1, x, x^2\} = \beta$ and $\mathbb{R}^{2\times 2}$ with basis $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Define $T : P_2 \to \mathbb{R}^{2\times 2}$ by

$$T(f(x)) = \begin{bmatrix} f(1) & f'(1) \\ f''(1) & f'''(1) \end{bmatrix}$$

Find $[T]_{\beta\gamma}$.

✓**Problem 6** Let $\beta = \{(2,5), (-3,1)\}$. Find the coordinates of $v = (a,b)$ with respect to the the basis β. That is, find $[v]_\beta$.

✓**Problem 7** Find a parametrization of a plane which contains the point $P = (1,2,3)$ and the vectors $\vec{A} = \langle 1,0,1\rangle$ and $\vec{B} = \langle 0,1,-2\rangle$ (these vectors are tangent to the plane).

✓**Problem 8** Let $A = [v_1|v_2|v_3|v_4|v_5]$ and suppose $\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Given this data, answer the following questions:

(a.) is $S = \{v_1, v_2\}$ a linearly independent set? If not then provide a linear dependence of the vectors in S

(b.) is $S = \{v_1, v_3, v_5\}$ a linearly independent set? If not then provide a linear dependence of the vectors in S

(c.) state the basis of the column space of A and find the $\dim(Col(A))$.

(d.) derive a basis for the solution set of $Ax = 0$ where $x = (x_1, x_2, x_3, x_4, x_5)$ and find the $\dim(Null(A))$.

(e.) is $[1, 1, 1, 1, 1]$ in $Row(A)$? (provide a short calculation to support your claim)

✓**Problem 9** Suppose $x = s + 7t$ and $y = 2s + 3t$ and $z = 4s - t$ parametrize some space in $\mathbb{R}^3$.

(a.) what type of space is this (a point, line, plane, volume etc...) ?

(b.) find the cartesian equation(s) of the space. How many equations in x, y, z should we expect are required?

✓**Problem 10** Suppose $w_1 = (1, 2, 3, 0)$ and $w_2 = (0, -3, 2, 0)$ and $w_3 = (1, 1, 1, 1)$. Find an orthonormal basis for $W = \text{span}\{w_1, w_2, w_3\}$. Find the projection of (a, b, c, d) onto the subspace W; that is, calculate $\text{Proj}_W(a, b, c, d)$.

✓**Problem 11** Find the least square fit line which best approximates the data set $(1, 2), (2, 4), (3, 7)$ $(4, 1), (5, 10), (-1, 0)$. (do this by solving appropriate normal equations via technology or hand-to-hand combat)

✓**Problem 12** Let $w_1 = E_{11}, w_2 = E_{12} + E_{21}, w_3 = E_{22} + E_{11}$. Use the gram-schmidt algorithm to find an orthonormal basis for $W = \text{span}\{w_1, w_2, w_3\}$ given that we use the inner product $< A, B >= \text{Trace}(AB^T)$.

✓**Problem 13** Find values for a, b, c which make the circle $ax^2 + by^2 = c$ closest to the data set $(4, 1), (-1, 5), (-5, 2), (2, -6), (3, 3)$. (do this by solving appropriate normal equations via technology)

✓**Problem 14** Find the projection of (v, w, x, y, z) onto $W = span\{w_1, w_2, w_3\}$ where $w_1 = (1, 1, 1, 1, 1)$, $w_2 = (0, 0, 1, -1, 2)$ and $w_3 = (1, 1, 0, 0, 1)$ by the following methods:

1. apply the Gram-Schmidt procedure to the given basis for W and create and orthonormal basis with which you may form the projection formula (we usually did this)
2. apply the beatifully simple matrix formula for finding projections as I mentioned in lecture. This does not require Gram-Schmidt and you should use technology to compute the matrix inverses and products required for the formula. (remind me about this please, I do want to share the formula when the time is ripe)

✓**Problem 15** Find the best approximation of $f(x) = e^x \sin(x)$ on $[-1,1]$ in P_2. (use the Legendre polynomials we already normalized, you may use wolfram alpha etc... to peform the requisite integration, we assume $< f, g >= \int_{-1}^{1} f(x)g(x)dx$.

✓**Problem 16** Use the method of least squares and explicit matrix computation (which can be done with Matlab etc...) to find the equation of the plane which is closest to the points:

$$(1,2,3),\ (4,5,6),\ (1,1,1),\ (4,8,5),\ (0,1,1),\ (2,2,2).$$

✓**Problem 17** Shores' 3.1# 17 (this one requires you read page 148)

✓**Problem 18** Shores' 3.2# 11 (easy span question)

✓**Problem 19** Shores' 3.2# 13 (easy, but, slightly abstract span question)

✓**Problem 20** Shores' 3.2# 17 (a proof question, but, have no fear, you can do it)

✓**Problem 21** Show that $\{1, (x-a), (x-a)^2, \ldots, (x-a)^n\}$ is a linearly independent subset of the vector space of polynomials.

✓**Problem 22** Find the coordinates of $f(x) = x^3 + 2 \in P_3(\mathbb{R})$ with respect to the basis

$$\beta = \{1, x-2, (x-2)^2, (x-2)^3\}.$$

✓**Problem 23** Shores' 3.3# 5 (coordinates w.r.t. given basis)

✓**Problem 24** Shores' 3.3# 13 (complete the matrix basis)

✓**Problem 25** Shores' 3.5# 17 (direct sum of symmetric and antisymmetric matrices)

✓**Problem 26** Shores' 4.1# 18 (an identity for the dot-product, just algebra note $v \bullet w = v^T w$ expresses dot-product as matrix multiplication)

✓**Problem 27** Shores' 4.2# $3b$ and $4b$ (sorry, SHORES' notation & terminology not good here.)

✓**Problem 28** Shores' 4.2# 9 (equation of plane)

✓**Problem 29** Shores' 4.2# 15 (orthogonal identity)

✓**Problem 30** Shores' 4.2# 20 (orthogonal identity)

✓**Problem 31** Shores' 4.3# 3 (orthogonal basis)

✓**Problem 32** Shores' 4.3# 5 (orthogonl and unitary matrices)

Problem 33 Shores' 4.3# 6 (coordinates w.r.t. orthogonal or unitary bases)

Problem 1 Consider $W = \{A \in \mathbb{R}^{2\times2} \mid A^T = A\}$. Is W a subspace of $\mathbb{R}^{2\times2}$? If the answer is yes then find a basis for W and state the dimension of W. Suppose $A \in W$.

$$A = A^T \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow A \in W \text{ has form } A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Observe $A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Thus $A \in \text{span}\{\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\beta}\}$ and conversely it is clear $\text{span}(\beta) \subset W$ thus $W = \text{span}(\beta)$ which shows W is subspace. Furthermore

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

and we find β is L.I. Consequently β is a basis and we find $\dim(W) = 3$.

Problem 2 Suppose $W = span\{v_1, v_2, v_3\} \subseteq \mathbb{R}^n$ for $n > 3$. Is W a subspace? If the answer is yes then state the possible dimensions of W.

Th^{m}/ $\text{span}(S) \leq \mathbb{R}^n$.

We're given $W = \text{span}(S)$ thus $W \leq \mathbb{R}^n$.

$\dim(W) \in \{0, 1, 2, 3\}$.

($\dim W = 0$ if $v_1 = v_2 = v_3 = 0$, $\dim W = 1$ if $v_1 = kv_2 = lv_3 \neq$ etc...

Problem 3 [2pts] Let $\beta = \{1, x^2, x\}$ be the **ordered** basis for P_2. Find the coordinate vector of $f(x) = (x-3)^2$.

$$\begin{aligned} f(x) &= x^2 - 6x + 9 \\ &= 9 + x^2 - 6x \end{aligned} \Longrightarrow \boxed{[f(x)]_\beta = \begin{bmatrix} 9 \\ 1 \\ -6 \end{bmatrix}}$$

Problem 4 For A given below:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Find,

(a.) a basis for $Col(A)$

(b.) a basis for $Null(A)$; (basis for the solution set of $Ax = 0$)

(c.) solve $Ax = (1, 1, 1, 1)$ if possible.

(d.) extend the basis of $Col(A)$ to obtain a basis for $\mathbb{R}^4$.

(a.) $\boxed{\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ gives basis for $Col(A)$.

(b.)
$x_1 = -x_2$
$x_3 = 3x_4$
$x_5 = 0$
x_2, x_4 free

$$x = \begin{bmatrix} -x_2 \\ x_2 \\ 3x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

form basis of $Null(A)$.

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

$Ax = (1,1,1,1)$ has sol$^{\underline{n}}$ when

(c.) $(1,1,1,1) \in Col(A)$ iff $\exists\, a, b, c$ such that

$$a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a = 1 \\ b = 1 \\ b = 1 \\ c = 1 \end{matrix}$$

thus $Ax = (1,1,1)$ has sol$^{\underline{n}}$ $x_1 = 1$, $x_3 = 1$, $x_5 = 1$
where $x_2 = x_4 = 0$ (this is not a $\rightarrow\leftarrow$ to (b.)
notice we're solving $Ax = (1,1,1,1)$ not $Ax = 0$)
Thus $\boxed{x = (1, 0, 1, 0, 1)}$

(d.) In (c.) changing $(1,1,1,1)$ to $(1,1,0,1)$ makes $\underline{b=1,\ b=0}$ inconsistent
thus $\beta \cup \{(1,1,0,1)\}$ will work.
(many answers here.)

Problem 5 Consider P_2 with basis $\{1, x, x^2\} = \beta$ and $\mathbb{R}^{2\times 2}$ with basis $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Define $T : P_2 \to \mathbb{R}^{2\times 2}$ by

$$T(f(x)) = \begin{bmatrix} f(1) & f'(1) \\ f''(1) & f'''(1) \end{bmatrix}$$

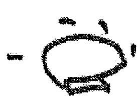
Find $[T]_{\beta\gamma}$.

$$T(\underbrace{a+bx+cx^2}_{f}) = \left.\left[\begin{array}{c|c} a+bx+cx^2 & b+2cx \\ \hline 2c & 0 \end{array}\right]\right|_{x=1} = \left[\begin{array}{c|c} a+b+c & b+2c \\ \hline 2c & 0 \end{array}\right]$$

$$\Rightarrow \quad [T(f)]_\gamma = \begin{bmatrix} a+b+c \\ b+2c \\ 2c \\ 0 \end{bmatrix}$$

we need $[T(f)]_\gamma = [T]_{\beta\gamma}[f]_\beta$

$$\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+2c \\ 2c \\ 0 \end{bmatrix}$$



$$\boxed{[T]_{\beta\gamma} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

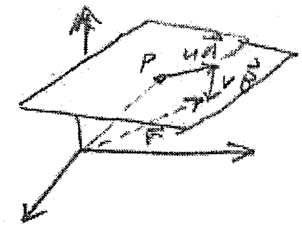
Problem 6 Let $\beta = \{(2,5), (-3,1)\}$. Find the coordinates of $v = (a,b)$ with respect to the the basis β. That is, find $[v]_\beta$.

$$[\beta] = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$$

$$[\beta]^{-1} = \frac{1}{2+15}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

$$\begin{aligned} [v]_\beta &= [\beta]^{-1} v \\ &= \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\ &= \boxed{\begin{bmatrix} \frac{1}{17}(a+3b) \\ \frac{1}{17}(-5a+2b) \end{bmatrix}} \end{aligned}$$

Problem 7. Find a parametrization of a plane which contains the point $P = (1,2,3)$ and the vectors $\vec{A} = \langle 1,0,1\rangle$ and $\vec{B} = \langle 0,1,-2\rangle$ (these vectors are tangent to the plane).



$$\vec{r}(u,v) = P + u\vec{A} + v\vec{B}$$

$$\boxed{\vec{r}(u,v) = \langle 1+u,\ 2+v,\ 3+u-2v\rangle}$$

Problem 8: [5pts] Let $A = [v_1|v_2|v_3|v_4|v_5]$ and suppose $\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Given this data, answer the following questions:

(a.) is $S = \{v_1, v_2\}$ a linearly independent set? If not then provide a linear dependence of the vectors in S

By CCP $V_2 = -2V_1$ hence not LI.

(b.) is $S = \{v_1, v_3, v_5\}$ a linearly independent set? If not then provide a linear dependence of the vectors in S

By CCP $V_5 = V_1 + V_3$ hence not LI.

(c.) state the basis of the column space of A and find the $\dim(Col(A))$.

The pivot columns of A are V_1, V_3 $\therefore$ $Col(A) = \text{span}\{V_1, V_3\}$ a nice basis is $\boxed{\{V_1, V_3\}}$ thus $\dim(Col(A)) = \boxed{2}$

(d.) derive a basis for the solution set of $Ax = 0$ where $x = (x_1, x_2, x_3, x_4, x_5)$ and find the $\dim(Null(A))$.

$$Ax = 0 \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{bmatrix} 2x_2 - 2x_4 - x_5 \\ x_2 \\ -3x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{W_1} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{W_2} + x_5 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{W_3}$$

Thus $\boxed{\{W_1, W_2, W_3\}}$ is basis for $Null(A)$ and $\boxed{\nu = 3}$

(e.) is $[1,1,1,1,1]$ in $Row(A)$? (provide a short calculation to support your claim)

$$a[1,-2,0,2,1] + b[0,0,1,3,1] = [1,1,1,1,1]$$

$$\left.\begin{aligned} a &= 1 \\ -2a &= 1 \\ &\vdots \end{aligned}\right\} \Rightarrow 1 = \frac{-1}{2} \quad \rightarrow\leftarrow$$

$\therefore$ no such $a, b \in \mathbb{R}$ exist hence $[1,1,1,1,1] \notin Row(A)$.

Problem 9 Suppose $x = s + 7t$ and $y = 2s + 3t$ and $z = 4s - t$ parametrize some space in $\mathbb{R}^3$.

(a.) what type of space is this (a point, line, plane, volume etc...) ?

this is a plane (containing $\langle 1,2,4\rangle$ & $\langle 7,3,-1\rangle$

$$\vec{r}(s,t) = s\langle 1,2,4\rangle + t\langle 7,3,-1\rangle$$

(b.) find the cartesian equation(s) of the space. How many equations in x, y, z should we expect are required?

Solve

$$x = s + 7t \quad \text{I} \quad \text{for } x, y, z.$$
$$y = 2s + 3t \quad \text{II}$$
$$z = 4s - t \quad \text{III}$$

take II $- 2$I to eliminate s

$$y - 2x = 3t - 14t = -11t \quad \text{IV}$$

Likewise III $- 2$II to eliminate s,

$$z - 2y = -t - 6t = -7t \quad \text{V}$$

Solve both IV and V for t,

$$t = \boxed{\frac{-1}{11}(y - 2x) = \frac{-1}{7}(z - 2y)}$$

((can clean up)

Problem 10 Suppose $w_1 = (1,2,3,0)$ and $w_2 = (0,-3,2,0)$ and $w_3 = (1,1,1,1)$. Find an orthonormal basis for $W = \text{span}\{w_1, w_2, w_3\}$. Find the projection of (a,b,c,d) onto the subspace W; that is, calculate $\text{Proj}_W(a,b,c,d)$.

Let $u_1 = \frac{1}{\sqrt{14}}\langle 1,2,3,0\rangle$ note $u_1 \cdot u_1 = 1$.

Let $v_2 = w_2 - (w_2 \cdot u_1)u_1 = (0,-3,2,0) - \frac{1}{14}(0)u_1 \Rightarrow u_2 = \frac{1}{\sqrt{13}}\langle 0,-3,2,0\rangle$

Let $v_3 = w_3 - (w_3 \cdot u_1)u_1 - (w_3 \cdot u_2)u_2$

$= (1,1,1,1) - \frac{1}{14}(1+2+3)\langle 1,2,3,0\rangle - \frac{1}{13}(-3+2)\langle 0,-3,2,0\rangle$

$= \langle 1,1,1,1\rangle - \frac{3}{7}\langle 1,2,3,0\rangle + \frac{1}{13}\langle 0,-3,2,0\rangle$

$= \langle 1 - 3/7,\ 1 - 6/7 - 3/13,\ 1 - 9/7 + 2/13,\ 1\rangle$

$= \langle 4/7,\ -8/91,\ -12/91,\ 1\rangle$

$\Rightarrow u_3 = \sqrt{\frac{91}{123}}\langle 4/7, -8/91, -12/91, 1\rangle$

Thus,

$\text{Proj}_W(a,b,c,d) = ((a,b,c,d)\cdot u_1)u_1 + ((a,b,c,d)\cdot u_2)u_2 + ((a,b,c,d)\cdot u_3)u_3$

$= \frac{1}{14}(a+2b+3c)\langle 1,2,3,0\rangle + \frac{1}{13}(-3b+2c)\langle 0,-3,2,0\rangle + \rightarrow$

$\rightarrow + \frac{91}{123}\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right)\langle \frac{4}{7}, \frac{-8}{91}, \frac{-12}{91}, 1\rangle$

Problem 2 [3pts] Find the least square fit line which best approximates the data set $(1,2), (2,4), (3,7)$ $(4,1), (5,10), (-1,0)$. (do this by solving appropriate normal equations via technology or hand-to-hand combat)

$$= \boxed{\begin{bmatrix} \frac{1}{14}a + \frac{2}{14}b + \frac{3}{14}c + \frac{4}{7}\left(\frac{91}{123}\right)\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \\ \frac{2}{14}a + \frac{4}{14}b + \frac{6}{14}c - \frac{3}{13}(-3b+2c) - \frac{8}{91}\left(\frac{91}{123}\right)\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \\ \frac{6}{14}a + \frac{6}{14}b + \frac{9}{14}c + \frac{2}{13}(-3b+2c) - \frac{12}{91}\left(\frac{91}{123}\right)\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \\ \frac{91}{123}\left(\frac{4a}{7} - \frac{8b}{91} - \frac{12c}{91} + d\right) \end{bmatrix}}$$

Alternatively, and computationally easier, let $A = [w_1 | w_2 | w_3]$

$\text{Proj}_W(v) = A(A^TA)^{-1}A^Tv$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

← can calculate via Matlab.

$\boxed{\text{PROBLEM 11}}$ find least squares line $y = ax + b$ for data set $(1,2), (2,4), (3,7), (4,1), (5,10), (-1,0)$.

Plug in the data to obtain

$$\begin{aligned} 2 &= a+b \\ 4 &= 2a+b \\ 7 &= 3a+b \\ 1 &= 4a+b \\ 10 &= 5a+b \\ 0 &= -a+b \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 4 \\ 7 \\ 1 \\ 10 \\ 0 \end{bmatrix}}_{b}$$

Hence, $$A^TA = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ -1 & 1 \end{bmatrix} = \left[\begin{array}{c|c} 56 & 14 \\ \hline 14 & 6 \end{array}\right]$$

$$A^Tb = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \\ 1 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 85 \\ 24 \end{bmatrix}$$

Normal Eq^{n}'s are $$\begin{bmatrix} 56 & 14 \\ 14 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 85 \\ 24 \end{bmatrix}$$

Thus $$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{6(56) - 14^2} \begin{bmatrix} 6 & -14 \\ -14 & 56 \end{bmatrix} \begin{bmatrix} 85 \\ 24 \end{bmatrix} = \begin{bmatrix} 87/70 \\ 11/10 \end{bmatrix} \therefore \boxed{y = \frac{87}{70}x + \frac{11}{10}}$$

aka $\boxed{y \cong 1.243x + 1.1}$

Problem 12 Let $w_1 = E_{11}, w_2 = E_{12} + E_{21}, w_3 = E_{22} + E_{11}$. Use the gram-schmidt algorithm to find an orthonormal basis for $W = \text{span}\{w_1, w_2, w_3\}$ given that we use the inner product $< A, B >= \text{Trace}(AB^T)$.

$$\langle W_1, W_1 \rangle = \text{Trace}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.$$

$$\langle W_2, W_1 \rangle = \text{Tr}\left((E_{12} + E_{21})E_{11}\right) = \text{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.$$

$$\langle W_2, W_2 \rangle = \text{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2.$$

$$\langle W_3, W_3 \rangle = \text{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$$

$$\langle W_2, W_3 \rangle = \text{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0.$$

$$\langle W_1, W_3 \rangle = \text{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.$$

Set $U_1 = W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ already normalized.

Set
$$V_2 = W_2 - \langle W_2, U_1 \rangle U_1$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 0\,U_1$$
$$= \underline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \Rightarrow \underline{U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}.$$

Next,
$$V_3 = W_3 - \langle W_3, U_1 \rangle U_1 - \underbrace{\langle W_3, U_2 \rangle}_{=0} U_2$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \text{Trace}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) U_2$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{note } \langle V_3, V_3 \rangle = 1$$

$$\boxed{\text{Thus } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ forms } \langle , \rangle\text{-orthonormal basis for } W.}$$

PROBLEM 13 find a, b, c such that

$ax^2 + by^2 = c$

closest to $(4,1)$, $(-1,5)$, $(-5,2)$, $(2,-6)$, $(3,3)$

Remark: grader, please skip this one.

$$16a + b = c$$
$$a + 25b = c$$
$$25a + 4b = c$$
$$4a + 36b = c$$
$$9a + 9b = c$$

$$\longrightarrow \underbrace{\begin{bmatrix} 16 & 1 & -1 \\ 1 & 25 & -1 \\ 25 & 4 & -1 \\ 4 & 36 & -1 \\ 9 & 9 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

well, this ~~allows~~ forces $a=b=c=0$ as sol$^{\underline{n}}$ to $A^TAv = 0$. Moreover, $Av = 0$ is consistent with $v=0$. So... should look at this differently.

$$\underbrace{\frac{a}{c}}x^2 + \underbrace{\frac{b}{c}}y^2 = 1 \quad \text{might do better,}$$

$$\alpha x^2 + \beta y^2 = 1$$

one of many interpretations. I don't think we have tools here to choose best route... Sorry for this problem ☺

$$16\alpha + \beta = 1$$
$$\alpha + 25\beta = 1$$
$$25\alpha + 4\beta = 1$$
$$4\alpha + 36\beta = 1$$
$$9\alpha + 9\beta = 1$$

$$\longrightarrow \underbrace{\begin{bmatrix} 16 & 1 \\ 1 & 25 \\ 25 & 4 \\ 4 & 36 \\ 9 & 9 \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{f}$$

$$A^TA = \left[\begin{array}{c|c} 979 & 366 \\ \hline 366 & 2019 \end{array}\right] \quad \text{and} \quad A^Tf = \begin{bmatrix} 55 \\ 75 \end{bmatrix}$$

$$\text{rref}\left[\begin{array}{cc|c} 979 & 366 & 55 \\ 366 & 2019 & 75 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & \frac{5573}{122,843} \\ 0 & 1 & \frac{3553}{122,843} \end{array}\right] \longrightarrow \begin{array}{l} \alpha \approx 0.04537 \\ \beta \approx 0.02892 \end{array}$$

$$\therefore \quad \boxed{0.04537x^2 + 0.02892y^2 = 1}$$

MATH 221

Show your work. Box answers please. If you use technology then explain how, do not just write answer. Thanks. You may work together, however, you must state who you worked with and you must write the answer in your own words in the end. Partial credit is given more generously to those who work alone. I would like the answers written on this page with the work attached. Thanks.

Problem 14 Find the projection of (v, w, x, y, z) onto $W = span\{w_1, w_2, w_3\}$ where $w_1 = (1,1,1,1,1)$, $w_2 = (0,0,1,-1,2)$ and $w_3 = (1,1,0,0,1)$ by the following methods:

1. apply the Gram-Schmidt procedure to the given basis for W and create and orthonormal basis with which you may form the projection formula (we usually did this)

$$\mathrm{Proj}_W \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3v + 3w - x + y + z \\ 3v + 3w - x + y + z \\ -v - w + 5x + 2y + 2z \\ v + w + 2x + 5y - 2z \\ v + w + 2x - 2y + 5z \end{bmatrix}$$

2. apply the beatifully simple matrix formula for finding projections as I mentioned in the last lecture. This does not require Gram-Schmidt and you should use technology to compute the matrix inverses and products required for the formula.

See my sol^n for decimal approx. of the f-la for $\mathrm{Proj}_W(v,w,x,y,z)$.

Problem 15 Find the best approximation of $f(x) = e^x \sin(x)$ on $[-1, 1]$ in P_2. (use the Legendre polynomials we already normalized, you may use wolfram alpha etc... to peform the requisite integration, we assume $< f, g >= \int_{-1}^{1} f(x)g(x)dx$.

$e^x \sin x \approx \cancel{0.273} + \cancel{0.7901x} + \cancel{0.1765x^2}$

$\underline{e^x \sin(x) \approx 0.3213 + 1.185x + 0.0314x^2}$

(work to follow in a few pages)

Problem 16 Use the method of least squares and explicit matrix computation (which can be done with Matlab etc...) to find the equation of the plane which is closest to the points:

$$(1,2,3),\ (4,5,6),\ (1,1,1),\ (4,8,5),\ (0,1,1),\ (2,2,2).$$

$\underline{z = 0.9383x + 0.1364y + 0.9616}$

(work shown in a few pages)

~~**Problem 4** [10pts] Suppose $dx/dt = x + y + z$ and $dy/dt = x - y + z$ and $dz/dt = x + 2z$. If the solution is at $(1,2,3)$ at $t = 0$ then find x, y, z as functions of t. Your solution may either use the built-in Matlab commands or the general method of solution by eigenvectors and matrix exponential as presented in my notes. Of course you can use technology to compute the eigenvalues of the matrix in question, it is probably not pretty.~~

$\cancel{x(t) = -8.09\exp(2.81t) + 11.29\exp(0.53t) - 2.19\exp(-1.34t)}$

$\cancel{y(t) = -4.8\exp(2.81t) + 2.37\exp(0.53t) + 4.43\exp(-1.34t)}$

$\cancel{z(t) = 10.01\exp(2.81t) - 7.67\exp(0.53t) + 0.65\exp(-1.34t)}$

See sol^n

PROBLEM 4

$$W_1 = (1, 1, 1, 1, 1) \longrightarrow \underline{\underline{u_1 = \frac{1}{\sqrt{5}} \langle 1, 1, 1, 1, 1 \rangle}}.$$

$$W_2 = (0, 0, +1, -1, 2)$$

$$W_3 = (1, 1, 0, 0, 1)$$

$$V_2 = W_2 - (W_2 \cdot u_1) u_1 = \langle 0, 0, 1, -1, 2 \rangle - \frac{2}{5} \langle 1, 1, 1, 1, 1 \rangle$$

$$\Rightarrow V_2 = \langle -2/5, -2/5, 3/5, -7/5, 8/5 \rangle$$

$$\Rightarrow V_2 = \frac{1}{5} \langle -2, -2, 3, -7, 8 \rangle$$

$$\Rightarrow u_2 = \frac{1}{\sqrt{4+4+9+49+64}} \langle -2, -2, 3, -7, 8 \rangle$$

$$\Rightarrow \underline{\underline{u_2 = \frac{1}{\sqrt{130}} \langle -2, -2, 3, -7, 8 \rangle}}.$$

$$\frac{-4}{130}$$

$$V_3 = W_3 - (W_3 \cdot u_1) u_1 - (W_3 \cdot u_2) u_2$$

$$= (1, 1, 0, 0, 1) - \frac{1}{5}(3) \langle 1, 1, 1, 1, 1 \rangle - \frac{1}{130}(-2-2+8) \langle -2, -2, 3, -7, 8 \rangle$$

$$= \langle 1, 1, 0, 0, 1 \rangle - \langle 3/5, 3/5, 3/5, 3/5, 3/5 \rangle + \left\langle \frac{8}{130}, \frac{8}{130}, \frac{-12}{130}, \frac{28}{130}, \frac{-32}{130} \right\rangle$$

$$= \left\langle 1 - \frac{3}{5} + \frac{8}{130}, 1 - \frac{3}{5} + \frac{8}{130}, \frac{-3}{5} - \frac{12}{130}, \frac{-3}{5} + \frac{28}{130}, 1 - \frac{3}{5} - \frac{32}{130} \right\rangle$$

$$= \langle 6/13, 6/13, -9/13, -5/13, 2/13 \rangle$$

$$= \frac{1}{13} \langle 6, 6, -9, -5, 2 \rangle$$

Normalizing yields $\underline{\underline{u_3 = \frac{1}{\sqrt{182}} \langle 6, 6, -9, -5, 2 \rangle}}.$

$$\mathrm{Proj}_W \underbrace{\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}}_{\vec{r}} = (u_1 \cdot \vec{r}) u_1 + (u_2 \cdot \vec{r}) u_2 + (u_3 \cdot \vec{r}) u_3 =$$

Problem 14 continued

$$\mathrm{Proj}_W(\vec{r}) = (u_1\cdot\vec{r})u_1 + (u_2\cdot\vec{r})u_2 + (u_3\cdot\vec{r})u_3 \qquad \text{where } \vec{r} = \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}$$

$$= \frac{1}{5}(v+w+x+y+z)\langle 1,1,1,1,1\rangle +$$

$$+ \frac{1}{130}(-2v-2w+3x-7y+8z)\langle -2,-2,3,-7,8\rangle +$$

$$+ \frac{1}{182}(6v+6w-9x-5y+2z)\langle 6,6,-9,-5,2\rangle$$

$$= \begin{bmatrix} \frac{1}{5}(v+w+x+y+z) - \frac{2}{130}(-2v-2w+3x-7y+8z) + \frac{6}{182}(6v+\cdots+2z) \\ \frac{1}{5}(v+w+x+y+z) - \frac{2}{130}(-2v-2w+3x-7y+8z) + \frac{6}{182}(6v+6w-9x-5y+2z) \\ \frac{1}{5}(v+w+x+y+z) + \frac{3}{130}(-2v-2w+3x-7y+8z) - \frac{9}{182}(6v+6w-9x-5y+2z) \\ \frac{1}{5}(v+w+x+y+z) - \frac{7}{130}(-2v-2w+3x-7y+8z) - \frac{5}{182}(6v+6w-9x-5y+2z) \\ \frac{1}{5}(v+w+x+y+z) + \frac{8}{130}(-2v-2w+3x-7y+8z) + \frac{2}{182}(6v+6w-9x-5y+2z) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{7}v + \frac{3}{7}w - \frac{1}{7}x + \frac{1}{7}y + \frac{1}{7}z \\ \frac{3}{7}v + \frac{3}{7}w - \frac{1}{7}x + \frac{1}{7}y + \frac{1}{7}z \\ -\frac{1}{7}v - \frac{1}{7}w + \frac{5}{7}x + \frac{2}{7}y + \frac{2}{7}z \\ \frac{1}{7}v + \frac{1}{7}w + \frac{2}{7}x + \frac{5}{7}y - \frac{2}{7}z \\ \frac{1}{7}v + \frac{1}{7}w + \frac{2}{7}x - \frac{2}{7}y + \frac{5}{7}z \end{bmatrix}$$

$$= \boxed{\frac{1}{7}\begin{bmatrix} 3v + 3w - x + y + z \\ 3v + 3w - x + y + z \\ -v - w + 5x + 2y + 2z \\ v + w + 2x + 5y - 2z \\ v + w + 2x - 2y + 5z \end{bmatrix}}$$

Problem 14. continued:

Use Eq. (10) on pg. 372 to avoid Gram-Schmidt,

$$\mathrm{Proj}_W(\vec{r}) = A\,(A^TA)^{-1}A^T\vec{r} \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Matlab calculates, to 4 decimals,

$$\mathrm{Proj}_W(\vec{r}) = \begin{bmatrix} 0.4286 & 0.4286 & -0.1429 & 0.1429 & 0.1429 \\ 0.4286 & 0.4286 & -0.1429 & 0.1429 & 0.1429 \\ -0.1429 & -0.1429 & 0.7143 & 0.2857 & 0.2857 \\ 0.1429 & 0.1429 & 0.2857 & 0.7143 & -0.2857 \\ 0.1429 & 0.1429 & 0.2857 & -0.2857 & 0.7143 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix}$$

Multiply this gives to a good approximation the same result as we found in PROBLEM 1.1.

$\frac{3}{7} \approx 0.4286$, $\frac{1}{7} \approx 0.1429$, $\frac{5}{7} \approx 0.7143$ etc...

PROBLEM 15 $\mathrm{Proj}_{P_2}(f(x)) = \sum_{j=1}^{3} \langle f(x), u_j \rangle u_j$ where $\langle u_i, u_j \rangle = \delta_{ij}$

and we calculated on pg. 266 of my notes)

$$u_1 = \frac{1}{\sqrt{2}}(1)$$

$$u_2 = \sqrt{\frac{3}{2}}\,x$$

$$u_3 = \sqrt{\frac{8}{45}}\left(x^2 - \frac{1}{3}\right)$$

PROBLEM [illegible].15 continued: thanks to Wolfram Alpha!

$$\langle f(x), u_1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} e^x \sin x \, dx \cong 0.4692$$

$$\langle f(x), u_2 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x e^x \sin x \, dx = 0.9677$$

$$\langle f(x), u_3 \rangle = \int_{-1}^{1} \sqrt{\frac{8}{45}} \left(x^2 - \frac{1}{3}\right) e^x \sin x \, dx = 0.0744.$$

Hence,

$$e^x \sin x \approx (0.4692)\frac{1}{\sqrt{2}} + (0.9677)\sqrt{\frac{3}{2}}\, x + 0.0744\sqrt{\frac{8}{45}}\left(x^2 - \frac{1}{3}\right)$$

$$\approx \boxed{0.3318 + \overset{1.185}{\cancel{0.7901}}\, x + \overset{0.0314}{\cancel{0.1765}}\left(x^2 - \frac{1}{3}\right)}.$$

$$\approx 0.3318 - \frac{1}{3}(0.1765) + 0.7901x + 0.1765x^2$$

$$= \boxed{\underset{0.3213}{\cancel{0.2730}} + \underset{1.185}{\cancel{0.7901}}\, x + \underset{0.0314}{\cancel{0.1765}}\, x^2}$$

~~PROBLEM 3~~ ~~ax+by+cz=d~~

$\boxed{\text{PROBLEM 16}}$ Find plane $z = ax + by + c$ closest to the points

$(1,2,3),\ (4,5,6),\ (1,1,1),\ (4,8,5),\ (0,1,1),\ (2,2,2)$

We try to solve:

$$\begin{aligned} 3 &= a + 2b + c \\ 6 &= 4a + 5b + c \\ 1 &= a + b + c \\ 5 &= 4a + 8b + c \\ 1 &= b + c \\ 2 &= 2a + 2b + c \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 1 & 1 & 1 \\ 4 & 8 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

The system $M\vec{v} = \vec{b}$ is inconsistent. We solve the normal eq$\underline{n}$'s $M^T M \vec{v} = M^T \vec{b}$ for the least-squares approx. sol$\underline{n}$ to the system $M\vec{v} = \vec{b}$.

$$\operatorname{rref}\left(M^T M \mid M^T \vec{b}\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.9383 \\ 0 & 1 & 0 & 0.1364 \\ 0 & 0 & 1 & 0.6916 \end{array}\right] \quad \text{by Matlab.}$$

$$\boxed{z = 0.9383x + 0.1364y + 0.6916}$$

PROBLEM 17 SHORES' §3.1 #17

Remark: didn't say nearly enough about homogeneous space to test in Spring 2017 semester

$$M = \left[\begin{array}{c|c} I_3 & \vec{t} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T_M(x_1, x_2, x_3, 1) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2 \\ x_2 - 1 \\ x_3 + 3 \\ 1 \end{bmatrix}$$

We see T_M moves the point $(x_1, x_2, x_3, 1)$ to $(x_1+2, x_2-1, x_3+3, 1)$ geometrically this is the <u>translation</u> of

$$(x_1, x_2, x_3) \longmapsto (x_1, x_2, x_3) + (2, -1, 3).$$

The inverse transformation is simply to translate back,

$$(x_1, x_2, x_3) \longmapsto (x_1, x_2, x_3) - (2, -1, 3)$$

Thus $T_M^{-1} = T_{M^{-1}}$ where $M^{-1} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$.

PROBLEM 18 SHORES' §3.2 #11

Show $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$

Well, if $(a,b) \in \mathbb{R}^2$ then $(a,b) = a(1,0) + b(0,1) \in \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ thus $\mathbb{R}^2 \subseteq \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and as $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^2$ is obvious we deduce $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$.

Likewise $(a,b) = c_1(1,0) + c_2(-2,1) = \begin{bmatrix} c_1 - 2c_2 \\ c_2 \end{bmatrix} \Rightarrow c_2 = b$ and $c_1 - 2b = a$ $\therefore c_1 = a + 2b$.

Thus $(a,b) = (a+2b)(1,0) + b(-2,1) \in \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$

thus it follows $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\} = \mathbb{R}^2$.

$$\therefore \quad \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$$

<u>Alternatively</u>: $\det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 1$ and $\det\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} = 1$

hence $\beta = \{(1,0), (0,1)\}$ and $\gamma = \{(1,0), (-2,1)\}$ have $\det[\beta] \neq 0$ & $\det[\gamma] \neq 0$

$\therefore$ β & γ are LI $\Rightarrow$ $\text{span}\,\beta = \text{span}\,\gamma = \mathbb{R}^2$.

P19 SHORES' §3.2 #13 / which ~~sets~~ spans below ~~span~~ give $P_2(\mathbb{R})$?

(a.) span $\{1, 1+x, x^2\}$ since $P_2(\mathbb{R}) = \text{span}\{1, x, x^2\}$ we know $P_2(\mathbb{R})$ has dimension $\underline{3}$ so if we can show $\{1, 1+x, x^2\}$ is LI then we know $\{1, 1+x, x^2\}$ forms a basis (because <u>all</u> LI sets of 3 vectors from basis for $P_2(\mathbb{R})$)

Consider,

$$c_1 + c_2(1+x) + c_3 x^2 = (c_1+c_2) + c_2 x + c_3 x^2 = 0$$

$$\Rightarrow \quad \left.\begin{array}{l} c_1 + c_2 = 0 \\ c_2 = 0 \end{array}\right\} \Rightarrow c_1 = 0$$

$$c_3 = 0$$

Thus $\{1, 1+x, x^2\}$ is LI and we deduce span$\{1, 1+x, x^2\} = P_2(\mathbb{R})$

<u>OR</u> $a + bx + cx^2 = cx^2 + b(x+1) - b + a$

$$= cx^2 + b(x+1) + (a-b)1 \in \text{span}\{1, 1+x, x^2\}$$

Hence span$\{1, 1+x, x^2\} = P_2(\mathbb{R})$ as $a+bx+cx^2$ is an arbitrary polynomial in $P_2(\mathbb{R})$ and I showed it is in the span of $\{1, x+1, x^2\}$.

(b.) span$\{x, 4x-2x^2, x^2\}$ is not $P_2(\mathbb{R})$ since

$4x - 2x^2 \in \text{span}\{x, x^2\}$ $\therefore$ span$\{x, x^2\} = \text{span}\{x, 4x-2x^2, x^2\}$

and clearly $1 \notin \text{span}\{x, x^2\}$.

(c.) span$\{1+x+x^2, 1+x, 3\}$

$$\begin{aligned} a + bx + cx^2 &= c(x^2+x+1) + bx - cx - c + a \\ &= c(x^2+x+1) + (b-c)(x+1) + c - b - c + a \\ &= c(x^2+x+1) + (b-c)(x+1) + \left(\frac{a-b}{3}\right)3 \end{aligned}$$

$\therefore$ $a+bx+cx^2 \in \text{span}\{x^2+x+1, 1+x, 3\}$

$$\Rightarrow \text{span}\{1+x+x^2, 1+x, 3\} = P_2(\mathbb{R}).$$

(d.) span$\{1-x^2, 1\} \neq P_2(\mathbb{R})$ since $\{1-x^2, 1\}$ only has 2 vectors we need 3 to span $P_2(\mathbb{R})$ (at least 3)

PROBLEM 20 SHORES' §3.2 #17

Let U, V be subspaces of W over $\mathbb{R}$.

(a.) $0 \in V$ and $0 \in U$ as $U, V \leq W$ thus $0 \in U \cap V \neq \emptyset$.
Next, suppose $x, y \in U \cap V$ then $x, y \in U$ and $x, y \in V$.
Let $c \in \mathbb{R}$ and observe $x, y \in U \Rightarrow cx + y \in U$.
Likewise, as $V \leq W$, $x, y \in V \Rightarrow cx + y \in V$.
Thus $cx + y \in U \cap V$ and we conclude $U \cap V$ is closed under vector addition & scalar multiplication.
Hence by subspace test Th^{m} we conclude $U \cap V \leq W$.

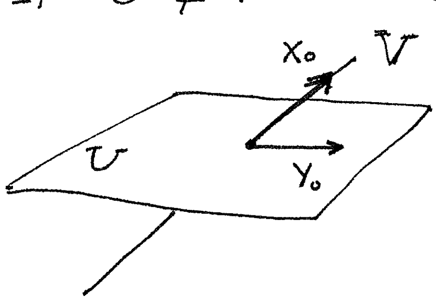
(b.) $U + V = \{ x + y \mid x \in U, \ y \in V \}$
Note $0 + 0 \in U + V \neq \emptyset$.
Let $x_1 + y_1, \ x_2 + y_2 \in U + V$ where $x_1, x_2 \in U$ and $y_1, y_2 \in V$
Let $c \in \mathbb{R}$ and observe as $U, V \leq W$ we find $cx_1 + x_2 \in U$
and $cy_1 + y_2 \in V$ consequently,

$$cx_1 + x_2 + cy_1 + y_2 = c(x_1 + y_1) + (x_2 + y_2) \in U + V$$

Consequently $U + V$ is closed under vect. add & scalar mult.
Hence $U + V \leq W$ by subspace test Th^{m}

(c.) If $U \subseteq V$ or $V \subseteq U$ then $U \cup V = V$ or $U \cup V = U$.
If $U \not\subseteq V$ then $\exists \, x_o \in V$ where $x_o \notin U$. Likewise

Remark: (c.) does NOT really FIT with instructions in SHORE ☹



if $V \not\subseteq U$ then $\exists \, y_o \in U$ with $y_o \notin V$.
Consider $x_o + y_o = v$.
Suppose $v \in U \cup V$ then
$\therefore$ $v \in U$ or $v \in V$. If $v = x_o + y_o \in V$ then $v - x_o = y_o$ and $v, x_o \in V \Rightarrow v - x_o \in V$ yet $y_o \notin V$ $\rightarrow\leftarrow$.
Likewise, if $v = x_o + y_o \in U$ then $v - y_o = x_o \notin V$
but $v, -y_o \in V \Rightarrow v - y_o \in V$ $\rightarrow\leftarrow$. Thus $v \notin U \cap V$
as every possibility produces $\rightarrow\leftarrow$. Hence $U \cup V$ not closed under $+$ $\therefore U \cup V \not\leq W$.

PROBLEM 21 Let $a \in \mathbb{R}$. Suppose

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = 0 \qquad (*)$$

Evaluate $(*)$ at $x=a$ to obtain $\underline{c_0 = 0}$. Differentiate $(*)$

$$c_1 + 2c_2(x-a) + \cdots + c_n n(x-a)^{n-1} = 0 \qquad (*)'$$

Evaluate $(*')$ at $x=a$ to obtain $\underline{c_1 = 0}$. Diff. w.r.t. x again,

$$2c_2 + 3\cdot 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2} = 0 \qquad (*)''$$

Eval. $(*'')$ at $x=a$ to obtain $2c_2 = 0 \therefore \underline{c_2 = 0}$. Continuing in this fashion we derive

$$c_0 = c_1 = \cdots = c_n = 0$$

Thus $\{1, x-a, (x-a)^2, \dots, (x-a)^n\}$ is LI.

PROBLEM 22 Let $\beta = \{1, x-2, (x-2)^2, (x-3)^3\}$ find $[v]_\beta$ for $v = x^3+2$

$$f(x) = x^3+2 = c_1 + c_2(x-2) + c_3(x-2)^2 + c_4(x-3)^3$$

$$f(2) = 8+2 = \underline{c_1 + c_4 = 10} \quad \text{I}$$

$$f'(2) = 3x^2\Big|_{x=2} = 12 = c_2 + 3c_4(2-3)^2 = \underline{c_2 + 3c_4 = 12} \quad \text{II}$$

$$f(0) = \underline{2 = c_1 - 2c_2 + 4c_3 - 27c_4} \quad \text{III}$$

$$f(3) = \underline{29 = c_1 + c_2 + c_3} \quad \text{IV}$$

Could solve I, II, III and IV for $[v]_\beta = (c_1, c_2, c_3, c_4)$.
But, I'll take another approach,

$$\beta = \{1, x-2, x^2-4x+4, x^3-9x^2+27x-27\}$$

Observe,

$$\begin{aligned} v = x^3+2 &= (x^3-9x^2+27x-27) + 9x^2 - 27x + 29 \\ &= (x^3-9x^2+27x-27) + 9(x^2+4x+4) + 36x - 36 - 27x + 29 \\ &= \underbrace{(x^3-9x^2+27x-27) + 9(x^2+4x+4) + 9(x-2) + 18 - 36 + 29}_{\text{read off } c_4=1,\ c_3=9,\ c_2=9,\ c_1=11} \end{aligned}$$

$$\therefore \boxed{[x^3+2]_\beta = (11, 9, 9, 1)}$$

Remark: it's really easy to miss this one by making dumb sign-error!

$\boxed{\text{PROBLEM 23}}$ SHORES' § 3.3 #5

(a.) $v = (-1,1)$ find $[v]_\beta$ for $\beta = \{(2,1), (2,-1)\}$

$$[v]_\beta = [\beta]^{-1} v = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{-1}{4} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\therefore \boxed{[v]_\beta = (1/4, -3/4)}$$

quick check: $\frac{1}{4}(2,1) - \frac{3}{4}(2,-1) = (1/2 - 6/4, \ 1/4 + 3/4) = (-1,1).$

(b.) $v = 2 + x^2$ find $[v]_\beta$ for $\beta = \{1+x, x+x^2, 1-x\}$

$$2 + x^2 = a(1+x) + b(x+x^2) + c(1-x)$$

$$2 + x^2 = a + c + (a+b-c)x + (b)x^2$$

$2 = a+c \qquad 0 = a+b-c \qquad 1 = b \implies$

$$\begin{aligned} a - c &= -1 \\ a + c &= 2 \end{aligned}$$

$2a = 1 \therefore a = 1/2.$

$\implies c = a + 1 = 3/2.$

$$\therefore \boxed{[v]_\beta = \left(\tfrac{1}{2}, 1, \tfrac{3}{2}\right)}$$

(c.) $v = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Hence for $\gamma = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ we find $\boxed{\left[\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right]_\gamma = (b, a, c)}$

(d.) $v = (1,2)$ find $[v]_\beta$ for $\beta = \{(2+i, 1), (-1, i)\}$ of $\mathbb{C}^2$

$$a(2+i, 1) \quad b(-1, i) = (1, 2)$$

$$((2+i)a - b, \ a + ib) = (1, 2)$$

$$\begin{bmatrix} 2+i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{i(2+i)+1} \begin{bmatrix} i & 1 \\ -1 & 2+i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2i} \begin{bmatrix} i + 2 \\ -1 + 2(2+i) \end{bmatrix} = \frac{-i}{2} \begin{bmatrix} 2+i \\ 3+2i \end{bmatrix}$$

$$\therefore [v]_\beta = \left(-i + 1/2, \ 1 - 3i/2\right)$$

P24 SHORES' § 3.3 #13

adjoin E_{ij} to $\left\{ \overset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}, \overset{B}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \overset{C}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}} \right\}$ to give basis for $\mathbb{R}^{2\times 2}$. Which E_{ij} work?

$$A - B = E_{22}$$

$$C - (A-B) = E_{21}$$

$$B - (C - A + B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{12}$$

Thus E_{11} is the answer by process of elimination ☺.

Better way? Use coordinates system $\{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$\left[[A]_\beta \,\middle|\, [B]_\beta \,\middle|\, [C]_\beta \,\middle|\, \begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix} \right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & 1 & 1 & c \\ 1 & 0 & 1 & d \end{array} \right]$$ then row

reduce this matrix to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & * \end{array} \right]$ and deduce the answer (I leave details of this 2nd sol$^{\underline{n}}$ to reader)

P25 SHORES' 3.5 #17

$$U = \{ A \in \mathbb{R}^{3\times 3} \mid A^T = A \} \;\&\; V = \{ A \in \mathbb{R}^{3\times 3} \mid A^T = -A \}$$

(a.) If $A \in U \cap V$ then $A^T = A$ and $A^T = -A \therefore A = -A \Rightarrow \underline{A = 0}$.
If $A \in \mathbb{R}^{3\times 3}$ then notice $\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A^T + A^{TT}) = \frac{1}{2}(A + A^T) \in U$
and $\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) \in V$ thus

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T) \in U + V. \Rightarrow \underline{\mathbb{R}^{3\times 3} = U + V}.$$

(b.) $V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \,\middle|\, \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \right\}$ $\longrightarrow a = 0$, $b = -d$; $\longrightarrow e = 0$, $c = -g$; $\longrightarrow i = 0$, $f = -h$

$\therefore A \in V$ has form $\begin{bmatrix} 0 & d & g \\ -d & 0 & h \\ -g & -h & 0 \end{bmatrix} = d\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{V_1} + g\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{V_2} + h\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{V_3}$

$\{V_1, V_2, V_3\}$ forms basis for $V \therefore \boxed{\dim V = 3}$

P26 Show $(A^T u) \cdot v = u \cdot (Av)$ for $A \in \mathbb{R}^{n \times n}$, $u, v \in \mathbb{R}^n$

$$(A^T u) \cdot v = (A^T u)^T v \quad : \text{writing dot-product as matrix mult.}$$
$$= u^T (A^T)^T v \quad : \text{socks-shoes identity}$$
$$= u^T A v$$
$$= u \cdot (Av).$$

P27 SHORES' §4.2 # 36 and 46

COMMENT: typo in SHORES' key to 36 also.

(36.) $u = (3,0,4)$, $v = (2,2,-1)$ find $\text{Proj}_v(u)$ and $\text{Comp}_v(u)$

$$\text{Proj}_v(u) = u - \left[\frac{(u \cdot v)}{v \cdot v}\right] v = (3,0,4) - \left(\frac{6-4}{9}\right)(2,2,-1)$$

$$\therefore \quad \text{Proj}_v(u) = (3 - 4/9,\ 4/9,\ 4 - 2/9)$$

$$\boxed{\text{Proj}_v(u) = (23/9,\ 4/9,\ 34/9)}$$

fine

But not what SHORE wants apparently.

Remark: my use of "projection" differs from that of SHORE. I would say find "vector component"

REMARK: Please ignore SHORE'S TERMINOLOGY ON PROJECTION AND ORTH. PROJECTION. STUDY OUR EXAMPLES FROM CLASS AND MY NOTES. SORRY.

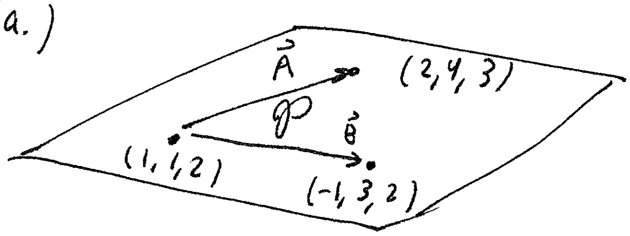
For us

$$\underbrace{\text{Proj}_{\vec{u}}(\vec{v}) = \vec{v} - \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u}}_{\text{projection of } \vec{v} \text{ onto the } \vec{u}\text{-direction}}$$

$$\text{Orth}_{\vec{u}}(\vec{v}) = \underbrace{\vec{v} - \text{Proj}_{\vec{u}}(\vec{v})}_{\text{orthogonal projection of } \vec{v} \text{ w.r.t. } \vec{u}\text{-direction.}}$$

P28 §4.2 #9 SHORE

(a.)



Various sol$^{\underline{n}}$'s exist. We need a normal vector to $\mathcal{P}$. Find $\vec{n} \in \{\vec{A}, \vec{B}\}^{\perp}$

$\vec{A} = \langle 1, 3, 1 \rangle$

$\vec{B} = \langle -2, 2, 0 \rangle$

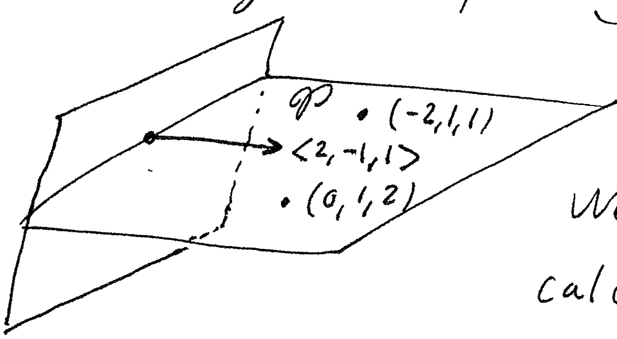
$$\begin{bmatrix} 1 & 3 & 1 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1/4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/4 \end{bmatrix}$$

Thus $\langle a, b, c \rangle \in \{\vec{A}, \vec{B}\}^{\perp}$ has $\begin{matrix} a = c/4 \\ b = c/4 \end{matrix}$

or $\langle c/4, \; c/4, \; c \rangle \Rightarrow \vec{n} = \langle 1, 1, 4 \rangle$ nice choice.

$\therefore \; (x-1) + (y-1) + 4(z-4) = 0$ or $\boxed{x + y + 4z = 18}$

(b.) plane with pts. $(-2,1,1)$ and $(0,1,2)$ which is orthogonal to plane $\underbrace{2x - y + z = 3}$



normal $\langle 2, -1, 1 \rangle$ must be tangent to $\mathcal{P}$

We seek normal to $\mathcal{P}$

calculate $\{\langle 2, 0, 1 \rangle, \langle 2, -1, 1 \rangle\}^{\perp}$

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \vec{n} = \langle -\tfrac{1}{2}c, 0, c \rangle$$

So select $\vec{n} = \langle -1, 0, 2 \rangle$ and use $(0,1,2)$ as base point for plane eq$^{\underline{n}}$, $-1(x-0) + 0(y-1) + 2(z-2) = 0$

$\Rightarrow \; -x + 2z - 4 = 0$

or $\boxed{x - 2z = -4}$

$\boxed{P29}$ §4.2 #15 / Let $u, v \in \mathbb{R}^n$,

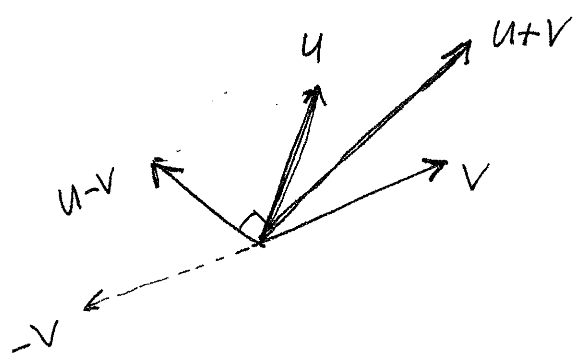
$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$$
$$= \|u\|^2 + 2u\cdot v + \|v\|^2$$

Thus $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $u\cdot v = 0$.

$\boxed{P30}$ §4.2 #20 / Let $\|u\| = \|v\| \Rightarrow$

$$(u+v)\cdot(u-v) = u\cdot u + \cancel{v\cdot u} - \cancel{u\cdot v} - v\cdot v = \|u\|^2 - \|v\|^2 = 0.$$

Hence $(u+v) \perp (u-v)$.



Geometrically

$\boxed{P32}$ oops.

(a.) $\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ∴ $\boxed{\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \text{ is orthogonal and it is its own inverse.}}$

(b.) $\overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}}^{R} \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}}^{R^T} = \frac{1}{2}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \neq I$ ∴ R not orthogonal

(c.) $\det \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{R} = 1(1+1) = 2$ but $R^T R = I \Rightarrow \det(R) = \pm 1$

thus $R^T R \neq I$

Hence $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ is not orthogonal matrix.

(d.)

P32 continued

(d.) $R = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

no need to calculate

$$RR^T = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & * & * & * \\ 0 & * & * & * \\ 2 & * & * & * \\ -2 & * & * & * \end{bmatrix}$$

give us $RR^T \neq I$ $\therefore$ R not orthogonal.

(e.) $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & i\sqrt{2} & 0 \\ i & 0 & -i \end{bmatrix}$

$$U^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -i\sqrt{2} & 0 \\ -i & 0 & i \end{bmatrix}^T = \boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -i \\ 0 & -i\sqrt{2} & 0 \\ 1 & 0 & i \end{bmatrix} = U^{-1}} \quad (*)$$

$$U^\dagger U = \frac{1}{2}\begin{bmatrix} 1 & 0 & -i \\ 0 & -i\sqrt{2} & 0 \\ 1 & 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & i\sqrt{2} & 0 \\ i & 0 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = I$$

Hence U is a unitary matrix and (*) gives $U^{-1} = U^\dagger$.

(f.) $\frac{1}{\sqrt{3}}\left[\begin{array}{c|c} 1+i & i \\ \hline i & 1-i \end{array}\right] = U \implies U^\dagger = \frac{1}{\sqrt{3}}\left[\begin{array}{c|c} 1-i & -i \\ \hline -i & 1+i \end{array}\right]$

$$U^\dagger U = \frac{1}{3}\left[\begin{array}{c|c} 1-i & -i \\ \hline -i & 1+i \end{array}\right]\left[\begin{array}{c|c} 1+i & i \\ \hline i & 1-i \end{array}\right] = \frac{1}{3}\left[\begin{array}{c|c} 2+1 & i+1-i+i^2 \\ \hline -i+1+i-1 & 1+2 \end{array}\right]$$

Hence $U^\dagger U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore$ U is unitary and

$$\boxed{U^{-1} = U^\dagger = \frac{1}{\sqrt{3}}\left[\begin{array}{c|c} 1-i & -i \\ \hline -i & 1+i \end{array}\right]}$$

P31 §4.3 #3

$$\beta = \{ \underbrace{(1,1,0)}_{V_1}, \underbrace{(-1,1,1)}_{V_2}, \underbrace{(1/2, -1/2, 1)}_{V_3} \}$$

$V_1 \cdot V_1 = 2$
$V_2 \cdot V_2 = 3$
$V_3 \cdot V_3 = \frac{2}{4} + 1 = \frac{3}{2}$

Since $V_1 \cdot V_2 = V_1 \cdot V_3 = V_2 \cdot V_3 = 0 \Rightarrow \beta$ orthogonal set of 3 vectors hence a LI set of 3 vectors in $\mathbb{R}^3$

Thus β is orthogonal basis.

$\vec{A} \cdot V_3 = \frac{1}{2} - 1 - 2 = \frac{1}{2} - \frac{6}{2} = \frac{-5}{2}$

(a.) $$\underbrace{(1,2,-2)}_{\vec{A}} = \left(\frac{\vec{A} \cdot V_1}{V_1 \cdot V_1}\right) V_1 + \left(\frac{\vec{A} \cdot V_2}{V_2 \cdot V_2}\right) V_2 + \left(\frac{\vec{A} \cdot V_3}{V_3 \cdot V_3}\right) V_3$$

$$[\vec{A}]_\beta = \left(3/2, -1/3, \frac{-5/2}{3/2}\right) = \boxed{(3/2, -1/3, -5/3)}$$

(b.) $\vec{B} = (1,0,0)$

$\vec{B} \cdot V_1 = 1$
$\vec{B} \cdot V_2 = -1$
$\vec{B} \cdot V_3 = 1/2$

$$\Rightarrow \boxed{[\vec{B}]_\beta = (1/2, -1/3, 1/3)}$$

(c.) $\vec{C} = (4,-3,2)$

$\vec{C} \cdot V_1 = 1$
$\vec{C} \cdot V_2 = -4 - 3 + 2 = -5$
$\vec{C} \cdot V_3 = 2 + \frac{3}{2} + 2 = \frac{8}{2} + \frac{3}{2} = \frac{11}{2}$

$$[\vec{C}]_\beta = \left(\frac{\vec{C} \cdot V_1}{V_1 \cdot V_1}, \frac{\vec{C} \cdot V_2}{V_2 \cdot V_2}, \frac{\vec{C} \cdot V_3}{V_3 \cdot V_3}\right) = \boxed{(1/2, -5/3, 11/3)}$$

$\boxed{P\,33}$ SHORES' §4.3 # 6

I will do (a.), (e.) and (f.)

(a.) $(2,4) = v$ and $\beta = \left\{ \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}, \frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix} \right\}$ has

$[\beta]$ an orthogonal matrix with $[\beta]^T[\beta] = I$

hence $\underbrace{[v]_\beta = [\beta]^{-1}v}_{\text{generally true for } \mathbb{R}^n} = \underbrace{[\beta]^T v}_{\text{special to } \beta \text{ orthonormal}} = \frac{1}{5}\begin{bmatrix}3 & 4\\4 & -3\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}22\\-4\end{bmatrix}$

$\therefore \boxed{[v]_\beta = (22/5, -4/5)}$

(e.) $(i, -2, 1) = w$ and

$\gamma = \left\{ \frac{1}{\sqrt{2}}(1, 0, i), (0, i, 0), (1/\sqrt{2}, 0, -i/\sqrt{2}) \right\}$ has $[\gamma]^\dagger[\gamma] = I$

Generally, for $\mathbb{C}^n$, $[w]_\gamma = [\gamma]^{-1}w$ but... as $[\gamma]^\dagger = [\gamma]^{-1}$

we find $[w]_\gamma = [\gamma]^\dagger w = \begin{bmatrix}1 & 0 & -i\\0 & -i\sqrt{2} & 0\\1 & 0 & i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}i\\-2\\1\end{bmatrix}$

sorry.

$$\boxed{[w]_\gamma = \frac{1}{\sqrt{2}}(0, -2i\sqrt{2}, 2i) = (0, -2i, i\sqrt{2})}$$

(f.) $(1,2) = z$ for $\delta = \left\{ \frac{1}{\sqrt{3}}(1+i, i), \frac{1}{\sqrt{3}}(i, 1-i) \right\}$

$[z]_\delta = [\delta]^{-1}z = [\delta]^\dagger z = \frac{1}{\sqrt{3}}\left[\begin{array}{c|c}1-i & -i\\ \hline -i & 1+i\end{array}\right]\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1-i-2i\\-i+2+2i\end{bmatrix}$

$$\boxed{[z]_\delta = \left(\frac{1}{\sqrt{3}}(1-3i), \frac{1}{\sqrt{3}}(2+i)\right)}$$