

## CURL AND DIVERGENCE IN CURVED COORDINATES

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We found the gradients already; this will hurt a little. We begin with the divergence in cylindricals,  $(r, \theta, z)$ , we need a few preliminary facts

$$\begin{aligned} e_r &= \cos\theta \hat{i} + \sin\theta \hat{j} & x &= r \cos\theta \\ e_\theta &= -\sin\theta \hat{i} + \cos\theta \hat{j} & y &= r \sin\theta \\ e_z &= \hat{k} & z &= z \end{aligned}$$

Then from the chain rule we find

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z} = -r \sin\theta \frac{\partial}{\partial x} + r \cos\theta \frac{\partial}{\partial y} \end{aligned}$$

We need to solve these for  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$ .

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r \sin\theta & r \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \quad \text{note} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -r \sin\theta & r \cos\theta \end{pmatrix}^{-1} = \frac{1}{r \cos^2\theta + r \sin^2\theta} \begin{pmatrix} r \cos\theta & -\sin\theta \\ r \sin\theta & r \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r \cos\theta & -\sin\theta \\ r \sin\theta & r \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \Rightarrow \boxed{\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \end{aligned}}$$

Calculate,

$$\begin{aligned} \nabla \cdot \vec{F} &= (\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}) \cdot (F_r e_r + F_\theta e_\theta + F_z e_z) \\ &= [\hat{i} (\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}) + \hat{j} (\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}) + \hat{k} \frac{\partial}{\partial z}] \cdot \\ &\quad \cdot [\hat{i} (F_r \cos\theta - F_\theta \sin\theta) + \hat{j} (F_r \sin\theta - F_\theta \cos\theta) + \hat{k} F_z] \\ &= \left( \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) (F_r \cos\theta - F_\theta \sin\theta) + \\ &\quad + \left( \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \right) (F_r \sin\theta - F_\theta \cos\theta) + \frac{\partial F_z}{\partial z} \\ &= \underbrace{\cos\theta \frac{\partial}{\partial r} [F_r \cos\theta - F_\theta \sin\theta]}_{\text{(I)}} - \underbrace{\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} [F_r \cos\theta - F_\theta \sin\theta]}_{\text{(II)}} \\ &\quad + \underbrace{\sin\theta \frac{\partial}{\partial r} [F_r \sin\theta + F_\theta \cos\theta]}_{\text{(III)}} + \underbrace{\frac{\cos\theta}{r} \frac{\partial}{\partial \theta} [F_r \sin\theta + F_\theta \cos\theta]}_{\text{(IV)}} + \frac{\partial F_z}{\partial z} \end{aligned}$$

Continuing to find  $\nabla \cdot \vec{F}$  in cylindrical coordinates. We focus on the pieces ①, ②, ③, ④ one at a time. To begin we pull out constant pieces. Remember generally  $F_r$  and  $F_\theta$  functions of both  $r, \theta$ , and  $z$ .

$$\textcircled{1} = \cos^2 \theta \frac{\partial F_r}{\partial r} - \cos \theta \sin \theta \frac{\partial F_\theta}{\partial r} \quad \textcircled{1}$$

$$\textcircled{2} = -\frac{\sin \theta}{r} \left[ \frac{\partial F_r}{\partial \theta} \cos \theta - F_r \sin \theta - \frac{\partial F_\theta}{\partial \theta} - F_\theta \cos \theta \right]$$

$$= -\frac{\sin \theta \cos \theta}{r} \frac{\partial F_r}{\partial \theta} + \frac{\sin^2 \theta}{r} F_r + \frac{\sin^2 \theta}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} F_\theta \quad \textcircled{2}$$

$$\textcircled{3} = \sin^2 \theta \frac{\partial F_r}{\partial r} + \sin \theta \cos \theta \frac{\partial F_\theta}{\partial r} \quad \textcircled{1}$$

$$\textcircled{4} = \frac{\cos \theta}{r} \left[ \frac{\partial F_r}{\partial \theta} \sin \theta + F_r \cos \theta + \frac{\partial F_\theta}{\partial \theta} - F_\theta \sin \theta \right]$$

$$= \frac{\sin \theta \cos \theta}{r} \frac{\partial F_r}{\partial \theta} + \frac{\cos^2 \theta}{r} F_r + \frac{\cos^2 \theta}{r} \frac{\partial F_\theta}{\partial \theta} - \frac{\sin \theta \cos \theta}{r} F_\theta \quad \textcircled{2}$$

Now lets assemble  $\nabla \cdot \vec{F}$  given the above results, notice certain terms cancel,

$$\nabla \cdot \vec{F} = (\cos^2 \theta + \sin^2 \theta) \frac{\partial F_r}{\partial r} + \frac{1}{r} (\sin^2 \theta + \cos^2 \theta) F_r + \frac{\sin^2 \theta + \cos^2 \theta}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial F_r}{\partial r} + \frac{1}{r} F_r + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \quad (*)$$

$$= \frac{1}{r} \left[ r \frac{\partial F_r}{\partial r} + F_r + \frac{\partial F_\theta}{\partial \theta} + r \frac{\partial F_z}{\partial z} \right]$$

$$= \boxed{\frac{1}{r} \left[ \frac{\partial}{\partial r} (r F_r) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial}{\partial z} (r F_z) \right] = \nabla \cdot \vec{F}}$$

The last formula and (\*) are probably both useful. Notice that on  $\mathbb{R}^2$  for  $\vec{F} = F_r \hat{e}_r + F_\theta \hat{e}_\theta = F_1 \hat{i} + F_2 \hat{j}$  we will find the same result without the  $z$ -terms,

$$\boxed{\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}}$$

$$\text{Thm} / \nabla \cdot \vec{F} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} [\rho^2 F_\rho] + \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} [\sin \varphi F_\varphi] + \frac{1}{\rho \sin \varphi} \frac{\partial F_\theta}{\partial \theta}$$

where  $\vec{F} = F_\rho \hat{e}_\rho + F_\varphi \hat{e}_\varphi + F_\theta \hat{e}_\theta$  in spherical coordinates  $\rho, \varphi, \theta$   
 where  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi$

Proof: We'll convert the def<sup>c</sup>  $\nabla \cdot \vec{F} = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$  to sphericals.  
 Lets recall what we already know from earlier work, (\*) 365,

$$F_x = \cos \theta \sin \varphi F_\rho + \cos \theta \cos \varphi F_\varphi - \sin \theta F_\theta$$

$$F_y = \sin \theta \sin \varphi F_\rho + \sin \theta \cos \varphi F_\varphi + \cos \theta F_\theta$$

$$F_z = \cos \varphi F_\rho - \sin \varphi F_\varphi$$

We also calculated that

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \frac{1}{\rho} \hat{e}_\varphi \frac{\partial}{\partial \varphi} + \frac{\hat{e}_\theta}{\rho \sin \varphi} \frac{\partial}{\partial \theta} : (*) \text{ on } 367$$

$$= (\hat{i} \cos \theta \sin \varphi + \sin \theta \sin \varphi \hat{j} + \cos \varphi \hat{k}) \frac{\partial}{\partial \rho} : \text{using } (*) \text{ on } 364$$

$$\frac{1}{\rho} (\hat{i} \cos \theta \cos \varphi + \hat{j} \sin \theta \cos \varphi - \sin \varphi \hat{k}) \frac{\partial}{\partial \varphi}$$

$$\frac{1}{\rho \sin \varphi} (-\sin \theta \hat{i} + \cos \theta \hat{j}) \frac{\partial}{\partial \theta}$$

$$= \hat{i} \left[ \cos \theta \sin \varphi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \theta \cos \varphi \frac{\partial}{\partial \varphi} - \frac{\sin \theta}{\rho \sin \varphi} \frac{\partial}{\partial \theta} \right] \quad \text{I}$$

$$+ \hat{j} \left[ \sin \theta \sin \varphi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \sin \theta \cos \varphi \frac{\partial}{\partial \varphi} + \frac{\cos \theta}{\rho \sin \varphi} \frac{\partial}{\partial \theta} \right] \quad \text{II}$$

$$+ \hat{k} \left[ \cos \varphi \frac{\partial}{\partial \rho} - \frac{\sin \varphi}{\rho} \frac{\partial}{\partial \varphi} \right] \quad \text{III}$$

$$= \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

We could derive  $\star$  from the chain rule, I took an alternative route.

Now  $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$  and we may compute

$$\nabla \cdot \vec{F} = \text{I} F_x + \text{II} F_y + \text{III} F_z \text{ where } F_x, F_y, F_z \text{ are}$$

expressed in terms of  $\rho, \varphi, \theta$ . We'll work out the gory details

Proof Continued :  $\nabla \cdot \vec{F}$  is SPHERICAL COORDINATES

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$$\begin{aligned}
 \textcircled{I} F_x &= \cos\theta \sin\phi \frac{\partial}{\partial p} [\cos\theta \sin\phi F_p + \cos\theta \cos\phi F_\phi - \sin\theta F_\theta] \\
 &\quad + \frac{1}{p} \cos\theta \cos\phi \frac{\partial}{\partial \phi} [\cos\theta \sin\phi F_p + \cos\theta \cos\phi F_\phi - \sin\theta F_\theta] \\
 &\quad - \frac{\sin\theta}{p \sin\phi} \frac{\partial}{\partial \theta} [\cos\theta \sin\phi F_p + \cos\theta \cos\phi F_\phi - \sin\theta F_\theta] \\
 &= \cos^2\theta \sin^2\phi \frac{\partial F_p}{\partial p} + \cos^2\theta \sin\phi \cos\phi \frac{\partial F_\phi}{\partial p} - \cos\theta \sin\theta \sin\phi \frac{\partial F_\theta}{\partial p} \\
 &\quad + \frac{1}{p} \cos^2\theta \cos\phi [\cos\phi F_p + \sin\phi \frac{\partial F_p}{\partial \phi} - \sin\phi F_\phi + \cos\phi \frac{\partial F_\phi}{\partial \phi}] \cancel{- \frac{\cos\theta \sin\theta \cos\phi}{p} \frac{\partial F_\theta}{\partial \phi}} \\
 &\quad - \frac{\sin\theta}{p} [-\sin\theta F_p + \cos\theta \frac{\partial F_p}{\partial \theta}] - \frac{\sin\theta \cos\phi}{p \sin\phi} [-\sin\theta F_\phi + \cos\theta \frac{\partial F_\phi}{\partial \theta}] \\
 &\quad - \frac{\sin\theta}{p \sin\phi} [-\cos\theta F_\theta + \sin\theta \frac{\partial F_\theta}{\partial \theta}]
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{II} F_y &= \sin\theta \sin\phi \frac{\partial}{\partial p} [\sin\theta \sin\phi F_p + \sin\theta \cos\phi F_\phi + \cos\theta F_\theta] \\
 &\quad + \frac{\sin\theta \cos\phi}{p} \frac{\partial}{\partial \phi} [\sin\theta \sin\phi F_p + \sin\theta \cos\phi F_\phi + \cos\theta F_\theta] \\
 &\quad + \frac{\cos\theta}{p \sin\phi} \frac{\partial}{\partial \theta} [\sin\theta \sin\phi F_p + \sin\theta \cos\phi F_\phi + \cos\theta F_\theta] \\
 &= \sin^2\theta \sin^2\phi \frac{\partial F_p}{\partial p} + \sin^2\theta \sin\phi \cos\phi \frac{\partial F_\phi}{\partial p} + \sin\theta \cos\theta \sin\phi \frac{\partial F_\theta}{\partial p} \\
 &\quad + \frac{\sin^2\theta \cos\phi}{p} [\cos\phi F_p + \sin\phi \frac{\partial F_p}{\partial \phi} - \sin\phi F_\phi + \cos\phi \frac{\partial F_\phi}{\partial \phi}] \cancel{+ \frac{\cos\theta \sin\theta \cos\phi}{p} \frac{\partial F_\theta}{\partial \phi}} \\
 &\quad + \frac{\cos\theta}{p} [\cos\theta F_p + \sin\theta \frac{\partial F_p}{\partial \theta}] + \frac{\cos\theta \cos\phi}{p \sin\phi} [\cos\theta F_\phi + \sin\theta \frac{\partial F_\phi}{\partial \theta}] \\
 &\quad + \frac{\cos\theta}{p \sin\phi} [-\sin\theta F_\theta + \cos\theta \frac{\partial F_\theta}{\partial \theta}]
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{III} F_z &= \cos\phi \frac{\partial}{\partial p} [\cos\phi F_p - \sin\phi F_\phi] - \frac{\sin\phi}{p} \frac{\partial}{\partial \phi} [\cos\phi F_p - \sin\phi F_\phi] \\
 &= \cos^2\phi \frac{\partial F_p}{\partial p} - \sin\phi \cos\phi \frac{\partial F_\phi}{\partial p} + \frac{\sin^2\phi}{p} F_p - \frac{\sin\phi \cos\phi}{p} \frac{\partial F_p}{\partial \phi} + \cancel{2} \\
 &\quad + \frac{\sin\phi \cos\phi}{p} F_\phi + \frac{\sin^2\phi}{p} \frac{\partial F_\phi}{\partial \phi}
 \end{aligned}$$

Proof Continued: sum ①  $F_x + ② F_y + ③ F_z$  using appropriate trig-identity and cancellations,

$$\begin{aligned}
 \nabla \cdot \vec{F} &= \frac{\partial F_p}{\partial p} \left[ \cos^2 \theta \sin^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \phi \right] \\
 &+ \frac{\partial F_\phi}{\partial p} \left[ \cos^2 \theta \sin \phi \cos \phi + \sin^2 \theta \sin \phi \cos \phi - \sin \phi \cos \phi \right] \\
 &+ \frac{\partial F_\theta}{\partial p} \left[ -\cos \theta \sin \theta \sin \phi + \sin \theta \cos \theta \sin \phi \right] \\
 &+ \frac{\partial F_p}{\partial \phi} \left[ \frac{1}{p} \cos^2 \theta \cos \phi \sin \phi + \frac{1}{p} \sin^2 \theta \cos \phi \sin \phi - \frac{1}{p} \sin \phi \cos \phi \right] \\
 &+ \frac{\partial F_\phi}{\partial \phi} \left[ \frac{1}{p} \cos^2 \theta \cos^2 \phi + \frac{1}{p} \sin^2 \theta \cos^2 \phi + \frac{\sin^2 \phi}{p} \right] \\
 &+ \frac{\partial F_p}{\partial \theta} \left[ -\frac{\sin \theta \cos \theta}{p} + \frac{\sin \theta \cos \theta}{p} \right] \\
 &+ \frac{\partial F_\phi}{\partial \theta} \left[ -\frac{\sin \theta \cos \theta}{p} + \frac{\cos \theta \sin \theta}{p} \right] \\
 &+ \frac{\partial F_\theta}{\partial \theta} \left[ +\frac{\sin^2 \theta}{p \sin \phi} + \frac{\cos^2 \theta}{p \sin \phi} \right] \\
 &+ F_p \frac{1}{p} \left[ \cos^2 \theta \cos^2 \phi + \sin^2 \theta + \sin^2 \theta \cos^2 \phi + \cos^2 \theta + \sin^2 \phi \right] \\
 &+ F_\phi \frac{1}{p} \left[ -\cancel{\cos^2 \theta \cos \phi \sin \phi} + \frac{\sin^2 \theta \cos \phi}{\sin \phi} - \cancel{\sin^2 \theta \cos \phi \sin \phi} + \frac{\cos^2 \theta \cos \phi}{\sin \phi} + \sin \phi \cos \phi \right] \\
 &+ F_\theta \frac{1}{p} \left[ \frac{\sin \theta \cos \theta}{\sin \phi} - \frac{\sin \theta \cos \theta}{\sin \phi} \right] \\
 \\ 
 &= \left( \frac{\partial F_p}{\partial p} + \frac{2}{p} F_p \right) + \frac{1}{p} \frac{\partial F_\phi}{\partial \phi} + \frac{\cos \phi}{p \sin \phi} F_\phi + \frac{1}{p \sin \phi} \frac{\partial F_\theta}{\partial \theta}
 \end{aligned}$$

Then notice that

$$\frac{1}{p^2} \frac{\partial}{\partial p} [p^2 F_p] = \frac{2p}{p^2} F_p + \frac{p^2}{p^2} \frac{\partial F_p}{\partial p} = \frac{2}{p} F_p + \frac{\partial F_p}{\partial p}$$

$$\frac{1}{p \sin \phi} \frac{\partial}{\partial \phi} [\sin \phi F_\phi] = \frac{\cos \phi}{p \sin \phi} F_\phi + \frac{1}{p} \frac{\sin \phi}{\sin \phi} \frac{\partial F_\phi}{\partial \phi} = \frac{1}{p} \frac{\partial F_\phi}{\partial \phi} + \frac{\cos \phi}{p \sin \phi} F_\phi$$

$$\therefore \boxed{\nabla \cdot \vec{F} = \frac{1}{p^2} \frac{\partial}{\partial p} [p^2 F_p] + \frac{1}{p \sin \phi} \frac{\partial}{\partial \phi} [\sin \phi F_\phi] + \frac{1}{p \sin \phi} \frac{\partial F_\theta}{\partial \theta}}$$

$$\text{Th}^3 / \nabla \times \vec{F} = \left( \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) e_r + \left( \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) e_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} [r F_\theta] - \frac{\partial F_r}{\partial \theta} \right) e_z$$

where  $\vec{F} = F_r e_r + F_\theta e_\theta + F_z e_z$  in cylindrical coordinates.

Proof: the definition was given in CARTESIAN's, focus on  $e_r \cdot (\nabla \times \vec{F})$

$$\begin{aligned} (\nabla \times \vec{F}) \cdot e_r &= (\nabla \times \vec{F}) \cdot (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= \cos \theta (\nabla \times \vec{F}) \cdot \hat{i} + \sin \theta (\nabla \times \vec{F}) \cdot \hat{j} \\ &= \cos \theta [\partial_y F_z - \partial_z F_y] + \sin \theta [\partial_z F_x - \partial_x F_z] \\ &= \underbrace{\cos \theta \partial_y F_z}_{\textcircled{I}} - \underbrace{\cos \theta \partial_z F_y}_{\textcircled{II}} + \underbrace{\sin \theta \partial_z F_x}_{\textcircled{III}} - \underbrace{\sin \theta \partial_x F_z}_{\textcircled{IV}} \end{aligned}$$

We'll proceed to convert  $\textcircled{I}, \textcircled{II}, \textcircled{III}, \textcircled{IV}$  to cylindricals, need to convert the partial derivatives, we already have formulas for  $F_x, F_y, F_z$  in terms of  $F_r, F_\theta, F_z$  (see (\*) on 363) and likewise for  $\partial_x, \partial_y, \partial_z$  in terms of  $\partial_r, \partial_\theta, \partial_z$  on 375. Calculate then,

$$\textcircled{I} = \cos \theta [\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta] [F_z] = \cancel{\cos \theta \sin \theta \partial_r F_z} + \frac{\cos^2 \theta}{r} \partial_\theta F_z$$

$$\textcircled{II} = -\cos \theta [\partial_z] [F_r \sin \theta + F_\theta \cos \theta] = -\cancel{\cos \theta \sin \theta \partial_z F_r} - \cos^2 \theta \partial_z F_\theta$$

$$\textcircled{III} = \sin \theta [\partial_z] [F_r \cos \theta - F_\theta \sin \theta] = \cancel{\cos \theta \sin \theta \partial_z F_r} - \sin^2 \theta \partial_z F_\theta$$

$$\textcircled{IV} = -\sin \theta [\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta] [F_z] = -\cancel{\sin \theta \cos \theta \partial_r F_z} + \frac{\sin^2 \theta}{r} \partial_\theta F_z$$

We see some encouraging cancellations, we find,

$$(\nabla \times \vec{F}) \cdot e_r = \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}$$

One component down, two to go.

- Our strategy here is to work on the curl one component at a time, this makes the calculation more manageable.

Proof Continued for  $\nabla \times \vec{F}$  in polar coordinates, the  $\theta$ -component

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$$\begin{aligned}
 (\nabla \times \vec{F}) \cdot \mathbf{e}_\theta &= (\nabla \times \vec{F}) \cdot [-\sin\theta \hat{i} + \cos\theta \hat{j}] \\
 &= -\sin\theta (\nabla \times \vec{F}) \cdot \hat{i} + \cos\theta (\nabla \times \vec{F}) \cdot \hat{j} \\
 &= \underbrace{-\sin\theta \partial_y F_z}_{\text{I}} + \underbrace{\sin\theta \partial_z F_y}_{\text{II}} + \underbrace{\cos\theta \partial_z F_x}_{\text{III}} - \underbrace{\cos\theta \partial_x F_z}_{\text{IV}}
 \end{aligned}$$

Proceed as before, convert to cylindrical coordinates,

$$\begin{aligned}
 \text{I} &= -\sin\theta [\sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta] [F_z] = -\sin^2\theta \partial_r F_z - \frac{\sin\theta \cos\theta}{r} \partial_\theta F_z \\
 \text{II} &= \sin\theta [\partial_z] [F_r \sin\theta + F_\theta \cos\theta] = \sin^2\theta \partial_z F_r + \sin\theta \cos\theta \partial_z F_\theta \\
 \text{III} &= \cos\theta [\partial_z] [F_r \cos\theta - F_\theta \sin\theta] = \cos^2\theta \partial_z F_r - \sin\theta \cos\theta \partial_z F_\theta \\
 \text{IV} &= -\cos\theta [\cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta] [F_z] = -\cos^2\theta \partial_r F_z + \frac{\sin\theta \cos\theta}{r} \partial_\theta F_z
 \end{aligned}$$

After a few cancellations in  $\text{I} + \text{II} + \text{III} + \text{IV}$ ,

$$(\nabla \times \vec{F}) \cdot \mathbf{e}_\theta = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}$$

And now the last component,

$$\begin{aligned}
 (\nabla \times \vec{F}) \cdot \mathbf{e}_z &= \partial_x F_y - \partial_y F_x \\
 &= [\cos\theta \partial_r - \frac{1}{r} \sin\theta \partial_\theta] [F_r \sin\theta + F_\theta \cos\theta] - \\
 &\quad - [\sin\theta \partial_r + \frac{1}{r} \cos\theta \partial_\theta] [F_r \cos\theta - F_\theta \sin\theta] \\
 &= \cancel{[\sin\theta \cos\theta \partial_r F_r]}^{\text{I}} + \cos^2\theta \partial_r F_\theta \cancel{\partial_r F_\theta}^{\text{II}} - \\
 &\quad - \frac{1}{r} \sin^2\theta \partial_\theta F_r - \frac{1}{r} \sin\theta \cos\theta F_r \cancel{\partial_\theta F_r}^{\text{III}} + \frac{1}{r} \sin\theta \cos\theta \partial_\theta F_\theta + \frac{1}{r} \sin^2\theta F_\theta \\
 &\quad - \cancel{[\sin\theta \cos\theta \partial_r F_r]}^{\text{I}} - \sin^2\theta \partial_r F_\theta + \frac{1}{r} \cos^2\theta \partial_\theta F_r - \frac{1}{r} \cos\theta \sin\theta F_r \cancel{\partial_\theta F_r}^{\text{III}} \\
 &\quad + \frac{1}{r} \cos\theta \sin\theta \partial_\theta F_\theta - \frac{\cos^2\theta}{r} F_\theta \\
 &= \partial_r F_\theta - \frac{1}{r} \partial_\theta F_r + \frac{1}{r} F_\theta = \left[ \frac{1}{r} \frac{\partial}{\partial r} [r F_\theta] - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right] = (\nabla \times \vec{F}) \cdot \mathbf{e}_z
 \end{aligned}$$

So we find that

$$\nabla \times \vec{F} = [(\nabla \times \vec{F}) \cdot \mathbf{e}_r] \mathbf{e}_r + [(\nabla \times \vec{F}) \cdot \mathbf{e}_\theta] \mathbf{e}_\theta + [(\nabla \times \vec{F}) \cdot \mathbf{e}_z] \mathbf{e}_z$$

$$\nabla \times \vec{F} = \left( \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \mathbf{e}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} [r F_\theta] - \frac{\partial F_r}{\partial \theta} \right) \mathbf{e}_z$$

Bonus Point: Derive the formula for  $\nabla \times \vec{F}$  in Spherical coordinates, its on next page

# SUMMARY OF DIFFERENTIAL VECTOR CALCULUS

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This table is borrowed from Thomas' Calculus 10<sup>th</sup> ed. Notice that  $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$  are denoted  $\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\phi$  and  $\mathbf{e}_\rho$  is  $\mathbf{u}_\rho$ . We have derived much of what is here. You should be equipped to derive the formulas for the Laplacian  $\nabla^2 f$  given the calculations I've shown you in these notes.

## Vector Operator Formulas in Cartesian, Cylindrical, and Spherical Coordinates; Vector Identities

### Formulas for Grad, Div, Curl, and the Laplacian

	<b>Cartesian (<math>x, y, z</math>)</b> $\mathbf{i}, \mathbf{j}$ , and $\mathbf{k}$ are unit vectors in the directions of increasing $x, y$ , and $z$ . $F_x, F_y$ , and $F_z$ are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.	<b>Cylindrical (<math>r, \theta, z</math>)</b> $\mathbf{u}_r, \mathbf{u}_\theta$ , and $\mathbf{k}$ are unit vectors in the directions of increasing $r, \theta$ , and $z$ . $F_r, F_\theta$ , and $F_z$ are the scalar components of $\mathbf{F}(r, \theta, z)$ in these directions.	<b>Spherical (<math>\rho, \phi, \theta</math>)</b> $\mathbf{u}_\rho, \mathbf{u}_\phi$ , and $\mathbf{u}_\theta$ are unit vectors in the directions of increasing $\rho, \phi$ , and $\theta$ . $F_\rho, F_\phi$ , and $F_\theta$ are the scalar components of $\mathbf{F}(\rho, \phi, \theta)$ in these directions.
<b>Gradient</b>	$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{u}_\theta + \frac{\partial f}{\partial z} \mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r} \mathbf{u}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{u}_\phi + \frac{1}{\rho \sin \phi} \frac{\partial f}{\partial \theta} \mathbf{u}_\theta$
<b>Divergence</b>	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 F_\rho)$ + $\frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} (F_\phi \sin \phi) + \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta}$
<b>Curl</b>	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{1}{r} \mathbf{u}_r & \mathbf{u}_\theta & \frac{1}{r} \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & F_\theta & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{u}_\rho & \mathbf{u}_\phi & \mathbf{u}_\theta \\ \frac{\partial}{\rho^2 \sin \phi} & \frac{\partial}{\rho \sin \phi} & \frac{\partial}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho \sin \phi F_\theta \end{vmatrix}$
<b>Laplacian</b>	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial f}{\partial \rho} \right)$ + $\frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2}$

### Vector Triple Products

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

### Vector Identities for the Cartesian Form of the Operator $\nabla$

In the identities listed here,  $f(x, y, z)$  and  $g(x, y, z)$  are differentiable scalar functions and  $\mathbf{u}(x, y, z)$  and  $\mathbf{v}(x, y, z)$  are differentiable vector functions.

$$\nabla \cdot f\mathbf{v} = f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f = f\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla) f$$

$$\nabla \times f\mathbf{v} - f\nabla \times \mathbf{v} + \nabla f \times \mathbf{v}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u})$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{2} \nabla(\mathbf{v} \cdot \mathbf{v})$$

# PHYSICS' CONVENTIONS FOR VECTOR CALCULUS

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Probably some of you will encounter the other type of spherical & cylindrical coordinates. In my experience physicists will always use these conventions where we have the following dictionary

Math	Physics
$\rho$	$r = \sqrt{x^2 + y^2 + z^2}$
$\theta$	$\phi, 0 \leq \phi \leq 2\pi$
$\phi$	$\Theta, 0 \leq \Theta \leq \pi$
$r$	$s = \sqrt{x^2 + y^2}$

For your convenience I include a few results borrowed from David Griffith's EXCELLENT (!) text on Electromagnetics.

## SPHERICAL AND CYLINDRICAL COORDINATES

### Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

$$\begin{cases} \hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\ \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \\ \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \end{cases}$$

### Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{x} = \cos \phi \hat{s} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \phi \hat{s} + \cos \phi \hat{\phi} \\ \hat{z} = \hat{z} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

$$\begin{cases} \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \\ \hat{z} = \hat{z} \end{cases}$$

# PHYSICS' CONVENTIONS CONTINUED

There is nothing really new here, we just take what we did and change notation according to the dictionary. These formulas are taken from the cover of Griffith's EM text.

## VECTOR DERIVATIVES

**Cartesian.**  $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$ ;  $d\tau = dx dy dz$

$$\text{Gradient : } \nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence : } \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl : } \nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\text{Laplacian : } \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical.**  $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$ ;  $d\tau = r^2 \sin \theta dr d\theta d\phi$

$$\text{Gradient : } \nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\phi}$$

$$\text{Divergence : } \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned} \text{Curl : } \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ &\quad + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi} \end{aligned}$$

$$\text{Laplacian : } \nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$$

**Cylindrical.**  $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\phi} + dz \hat{\mathbf{z}}$ ;  $d\tau = s ds d\phi dz$

$$\text{Gradient : } \nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\phi} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence : } \nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (sv_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl : } \nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\phi} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (sv_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$\text{Laplacian : } \nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$