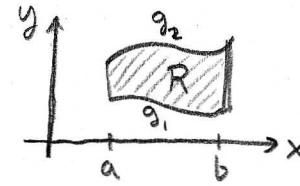


## DOUBLE INTEGRALS OVER GENERAL REGIONS:

(334)

Given an arbitrary connected region in the  $xy$ -plane there are two primary descriptions of the region, say  $R$  (not necessarily a rectangle any more). Your text classifies them as,

TYPE I:  $\begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases}$



TYPE II:  $\begin{cases} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{cases}$



Of course, you can imagine regions which don't conveniently fit either TYPE. And on the other hand a rectangle is both TYPES at once,  $g_1(x)=c$ ,  $g_2(x)=d$  to get TYPE I,  $h_1(y)=a$ ,  $h_2(y)=b$  to get TYPE II.

**Th<sup>n</sup> (FUBINI, STRONG VERSION):** Suppose  $f$  is mostly continuous.

Given  $R_I = \{f(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$  a TYPE I region,

$$\iint_{R_I} f(x,y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx.$$

Given  $R_{II} = \{f(x,y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$  a TYPE II region,

$$\iint_{R_{II}} f(x,y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy.$$

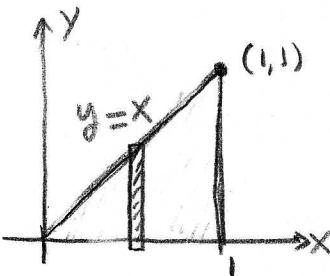
**E93** Let  $R = \{f(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ .

$$\begin{aligned} \iint_R e^{x^2} dA &= \int_0^1 \int_0^y e^{x^2} dy dx = \int_0^1 (e^{x^2} y \Big|_0^x) dx = \int_0^1 x e^{x^2} dx \\ &= \frac{1}{2} e^{x^2} \Big|_0^1 \end{aligned}$$

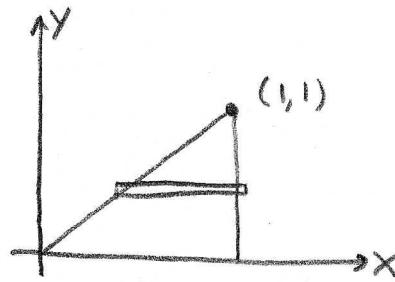
$$= \frac{1}{2}(e^1 - e^0) \\ = \boxed{\frac{1}{2}(e - 1)}$$

Remark: we could just as well describe  $R$  as a TYPE II region. However, then we'd be faced with  $\int e^{x^2} dx$ . This is not an elementary integral.

E94 Using R from E93 calc.  $\iint_R e^{y^2} dA$ . Since treating R as TYPE I leads us to  $\int_0^1 \int_{y^2}^y e^{y^2} dy dx$  we need to make dx come first, so convert R to a TYPE II region. A picture helps,



$$y_{\text{TOP}} = x \\ y_{\text{bottom}} = 0$$



$$x_{\text{LEFT}} = y \\ x_{\text{RIGHT}} = 1$$

$$\underbrace{\{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}}_{\text{TYPE I description}} = R = \underbrace{\{(x,y) \mid 0 \leq y \leq 1, y \leq x \leq 1\}}_{\text{TYPE II description}}$$

$$\begin{aligned} \iint_R e^{y^2} dA &= \int_0^1 \int_y^1 e^{y^2} dx dy : \quad \left[ \begin{array}{l} \text{the integral of the constant } e^{y^2} \\ \text{is the product of the integration} \\ \text{region length } (1-y) \text{ and the constant.} \end{array} \right] \\ &= \int_0^1 (1-y) e^{y^2} dy : \\ &= \int_0^1 e^{y^2} dy - \int_0^1 y e^{y^2} dy \\ &= \int_0^1 e^{y^2} dy - \frac{1}{2}(e-1) : \quad \text{curses, I had hoped for better.} \\ &= 1.463 - \frac{1}{2}(e-1) : \quad \text{} \int_0^1 e^{y^2} dy \text{ req's numerical soln.} \end{aligned}$$

Remark: Not all integrals result in pretty sums & products, if we just make up some example on a hunch then it can get ugly. Incidentally while indefinite integrals of  $e^{x^2}$  are not known in terms of elementary functions, there are improper integrals of  $e^{-x^2}$  which do come out quite nicely. See §16.4 #36, we need a few toys to make it easier.

**E95** Another application of double integrals is finding the area of a region. For example,  $S = \{(x, y) \mid 0 \leq x \leq R, 0 \leq y \leq \sqrt{R^2 - x^2}\}$

$$\begin{aligned}
 A(R) &= \iint_S dA = \int_0^R \int_0^{\sqrt{R^2-x^2}} dy dx \\
 &= \int_0^R \sqrt{R^2-x^2} dx && : \text{use trig-substitution } x = R\cos\theta \\
 &= \int_{\pi/2}^0 -R^2 \sin^2\theta d\theta && \text{so } dx = -R\sin\theta d\theta \text{ and} \\
 &= R^2 \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta) d\theta && \sqrt{R^2-x^2} = \sqrt{R^2\sin^2\theta} = R\sin\theta \text{ and} \\
 &= \frac{R^2}{2} \left( \theta - \frac{1}{2}\sin 2\theta \right) \Big|_0^{\pi/2} && \text{the bounds change to } \pi/2 \rightarrow 0 \\
 &= \frac{R^2}{2} \left( \frac{\pi}{2} \right) = \boxed{\pi R^2/4}. && : \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\
 &&& : \text{since } S \text{ is a quarter-circle} \\
 &&& \text{this makes good sense.}
 \end{aligned}$$

Remark: this will be much easier in polar coordinates, see **E106** on **344**.

**-76** We may also define the average of a function over  $R$  as

$$f_{\text{avg}}^R \equiv \frac{1}{A(R)} \iint_R f(x, y) dA$$

Consider  $f(x, y) = xy$ . If  $R = [0, 1] \times [0, 1]$  and  $S$  = quarter circle with  $R=1$  from E95, do you think  $f_{\text{avg}}^R > f_{\text{avg}}^S$  or vice-versa?

$$\iint_R xy dA = \int_0^1 \int_0^1 xy dx dy = \int_0^1 x dx \int_0^1 y dy = \frac{1}{2} \frac{1}{2} = \frac{1}{4}.$$

$$\iint_S xy dA = \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx = \int_0^1 \left( \frac{1}{2}xy^2 \Big|_0^{\sqrt{1-x^2}} \right) dx = \int_0^1 \frac{1}{2}(x - x^3) dx = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{1}{8}$$

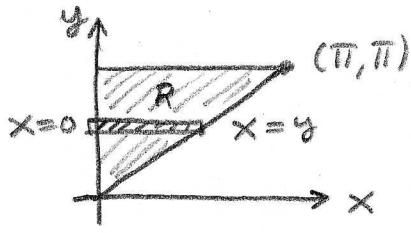
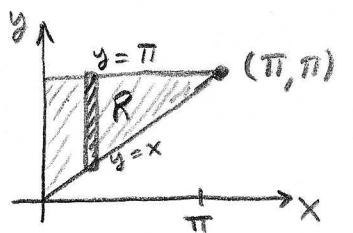
Thus

$$f_{\text{avg}}^R = \frac{1/4}{A(R)} = \frac{1/4}{1} = \frac{1}{4} \quad \text{whereas } f_{\text{avg}}^S = \frac{1/8}{A(S)} = \frac{1/8}{\pi/4} = \frac{1}{8\pi}$$

The average of  $xy$  is larger on the unit-square since  $\frac{1}{4} > \frac{1}{8\pi}$ .

Remark: Notice  $\iint_S xy dA$  was considerably easier than  $\iint_S dA$ .

E97) Calculate  $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy dx$ . Notice we need to reverse the order of integration to do a TYPE II integration (has  $dx dy$  instead). Our given integral suggests  $0 \leq x \leq \pi$  and  $x \leq y \leq \pi$ ,



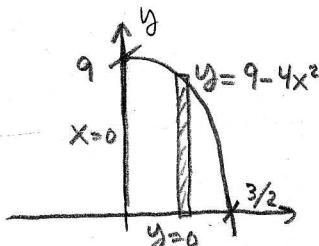
$$\underline{0 \leq y \leq \pi}, \underline{0 \leq x \leq y}$$

$$\iint_R \frac{\sin y}{y} dA = \int_0^{\pi} \int_0^y \frac{\sin y}{y} dx dy = \int_0^{\pi} \sin y dy = -\cos y \Big|_0^{\pi} = 2.$$

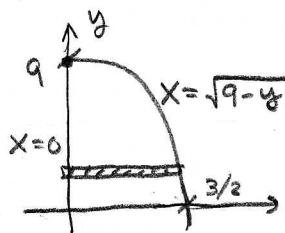
E98) Calculate,

$$\begin{aligned} \int_0^{3/2} \int_0^{9-4x^2} 16x dy dx &= \int_0^{3/2} 16x(9-4x^2) dx : 16x \text{ is a constant wrt } dy \text{ integration, simply multiply by int. reg. length.} \\ &= \int_0^{3/2} (144x - 64x^3) dx \\ &= 72x^2 \Big|_0^{3/2} - 16x^4 \Big|_0^{3/2} \\ &= 72(3/2)^2 - 16(3/2)^4 \\ &= 162 - 81 = 81 \end{aligned}$$

Let's reverse the order of integration for fun. Note  $9-4x^2=0 \Rightarrow x^2 = \frac{9}{4}$



TYPE I: find y bounds in terms of x



TYPE II: find x bounds in terms of y.

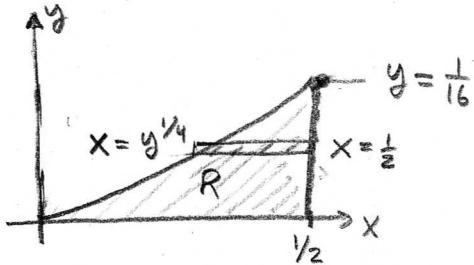
$y = 9-4x^2$  is a parabola with x-intercepts  $x = \pm \sqrt{9/4}$  and y-intercept 9. Solve for x and keep positive root,

$$x^2 = \frac{1}{4}(9-y)$$

$$x = \frac{1}{2}\sqrt{9-y}$$

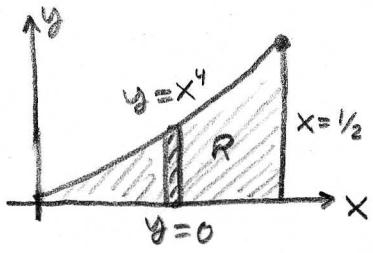
$$\begin{aligned} \iint_R 16x dA &= \int_0^9 \int_{\frac{1}{2}\sqrt{9-y}}^{3/2} 16x dx dy = \int_0^9 (8x^2 \Big|_{\frac{1}{2}\sqrt{9-y}}^{3/2}) dy = \int_0^9 2(9-y) dy \\ &= (18y - y^2 \Big|_0^9) \\ &= 18(9) - 81 \\ &= 81 \end{aligned}$$

E99 Calculate  $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) dx dy$ . Seems changing bounds may be helpful here. To begin  $0 \leq y \leq 1/16$  and  $y^{1/4} \leq x \leq 1/2$  which is type II, let's graph to guide our conversion to type I,



$$\frac{1}{2} = y^{1/4} \Rightarrow y = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

thus  $x = 1/2$  and  $x = y^{1/4}$  intersect at the point  $(y_2, y_{16})$  as graphed.



$$0 \leq x \leq 1/2$$

$$0 \leq y \leq \underline{\underline{x^4}}$$

well isn't that convenient.

$$\begin{aligned}
 \iint_R \cos(16\pi x^5) dA &= \int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5) dy dx : \text{ notice } \cos(16\pi x^5) \text{ is constant} \\
 &\quad \text{in the } dy - \text{integration.} \\
 &= \int_0^{1/2} x^4 \cos(16\pi x^5) dx : \text{ let } u = 16\pi x^5, \\
 &= \frac{1}{80\pi} \sin(16\pi x^5) \Big|_0^{1/2} \\
 &= \frac{1}{80\pi} \left( \sin\left(\frac{16\pi}{32}\right) - \sin(0) \right) \\
 &= \boxed{\frac{1}{80\pi}}
 \end{aligned}$$

Remark: these arguments should be familiar from Calc. II, see p. 132 - 134. Our methods for finding area were more specialized, now we add  $f(x, y)$  into the integration but the essential idea of viewing the graph as TYPE I or TYPE II was there as well. I'd say type II regions needed horizontal slicing whereas type I were vertically sliced. We can see the formula on 132 as type I

$$A = \int_a^b \int_{g(x)}^{f(x)} dy dx = \int_a^b (f(x) - g(x)) dx = \int_a^b (Y_{top} - Y_{bottom}) dx \leftarrow$$

$$\text{or for TYPE II: } A = \int_c^d \int_{X_L(y)}^{X_R(y)} dx dy = \int_c^d (X_R - X_L) dy \leftarrow \text{These formulas are special cases of our double integrals.}$$