

FUNDAMENTAL Th^m FOR LINE INTEGRALS (§17.3)

395

This theorem helps us understand the importance of the concept of the "conservative vector field". In E144, E145 we saw that certain vector fields could be written as the gradients of a scalar function. We defined earlier on 360 that \vec{F} is conservative if $\exists f$ such that $\vec{F} = \nabla f$. Then on 370 we found that if $\vec{F} = \nabla f$ then $\nabla \times \vec{F} = 0$. We say \vec{F} is irrotational iff $\nabla \times \vec{F} = 0$. Then on 371 we learned $\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$ for some f , in all cases. However, if $\text{dom}(\vec{F})$ is simply connected or all of \mathbb{R}^3 then $\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla f$ for some f . Given a conservative vector field \vec{F} we found how to calculate f such that $\vec{F} = \nabla f$. The little " f " is called a potential function for \vec{F} . Let's see what the integrals have to do with this story,

Th^m/ Let C be an oriented, smooth (nonstop) curve given by $\vec{r}(t)$ $a \leq t \leq b$. Let f be a differentiable function whose gradient vector ∇f is continuous on C . Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

"Fundamental Th^m for Line Integrals"

- We'll prove it after a few examples & comments.

Corollary: Given C as above and $\vec{F} = \nabla f$ then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

E148 in E145 on 393 we considered $\vec{F} = \langle 0, 0, -mg \rangle$. We found that $\int_C \vec{F} \cdot d\vec{r} = -mgh$ for the helix C which had starting point $\vec{r}(0) = \langle a, 0, 0 \rangle$ and ending point $\vec{r}(h) = \langle a \cosh h, a \sinh h, h \rangle$. Notice that the potential function for this \vec{F} is easy to find by inspection,

$$f = -mgz \quad \text{observe } \nabla f = \langle 0, 0, -mg \rangle = \vec{F}$$

Lets check and see if the FTC for line integrals works,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(h)) - f(\vec{r}(0)) = -mg \left. z \right|_{\vec{r}(0)}^{\vec{r}(h)} = -mgh$$

E149 we found in E130 on 370 that $\vec{F} = \langle 2x+y, 3\cos(yz)+x, y\cos(yz) \rangle$ has the potential function $f = x^2 + xy + \sin(yz)$. You can check that $\vec{F} = \nabla f$. Let C be any smooth (nonstop) curve from $(0,0,0)$ to $(1,1,\pi/2)$ find $\int_C \vec{F} \cdot d\vec{r}$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C \nabla f \cdot d\vec{r} = f(1,1,\pi/2) - f(0,0,0) \\ &= 1 + 1 + \sin(\pi/2) - 0 \\ &= 3\end{aligned}$$

- Notice the curve C was basically arbitrary. We could have taken another curve \tilde{C} and obtained $\int_{\tilde{C}} \vec{F} \cdot d\vec{r} = 3$. This independence of the curve taken is defined as follows,

Defn If $\int_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for all smooth oriented curves inside the domain of \vec{F} then we say that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, or \vec{F} is a path-independent.

Not all vector fields are path independent, for example we found that $\int_C \vec{F} \cdot d\vec{r} = \frac{1}{2}$ in E141 on 391 yet $\int_{C_2} \vec{F} \cdot d\vec{r} = \frac{41}{130}$ in E143 on 392. We say such a \vec{F} is path dependent.

- So we should prove the FTC for line integrals before we forget,
Proof: Suppose C has parametrization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b (\nabla f)(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \quad \text{multivariate chain rule.} \\ &= \int_a^b \frac{d}{dt} [f(x(t), y(t), z(t))] dt \\ &= \int_a^b \frac{d}{dt} [f(\vec{r}(t))] dt \quad : \text{just changing notation.} \\ &= f(\vec{r}(b)) - f(\vec{r}(a)). // \quad : \text{using the FTC of one variable calculus.}\end{aligned}$$

the proof is simple, as usual we simply borrow from calc. I to make the really nontrivial step.

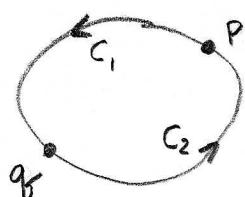
INDEPENDENCE OF PATH (§17.3)

We should generalize a bit and include piecewise smooth curves (which Stewart calls "paths", sorry there seems to be some disagreement as to what should be termed a path or curve.)

Thⁿ / $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D \Leftrightarrow \int_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths in D .

Proof: To prove \Leftrightarrow we must prove \Rightarrow and \Leftarrow .

\Rightarrow Suppose $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D . Let C be a closed path in D . $C = C_1 \cup C_2$ where



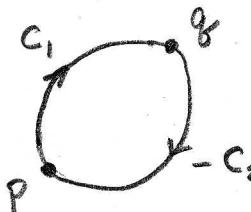
C_1 goes from P to Q and
 C_2 goes from Q to P thus
 $-C_2$ goes from P to Q .

path independence $\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r} = - \int_{C_2} \vec{F} \cdot d\vec{r}$ then,

$$0 = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \boxed{\int_C \vec{F} \cdot d\vec{r} = 0}$$

And C was an arbitrary closed curve so we have it for all closed curves in D .

\Leftarrow Let C_1, C_2 be two curves in D from P to Q then $-C_2$ goes from Q to P . Thus $C_1 \cup \{-C_2\}$ is a closed curve in D . Thus by assumption



$$\int_{C_1 \cup \{-C_2\}} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\therefore \boxed{\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \quad \forall C_1, C_2 \text{ in } D.}$$

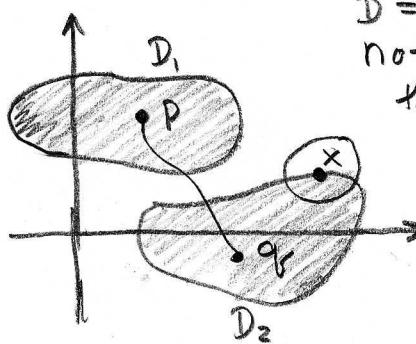
- the proof follows mainly from our result that reversing the orientation changes the sign, see (390) (*).

Independence of Path Continued

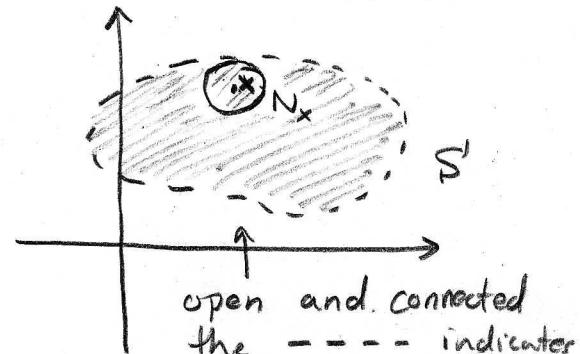
The proof of the Th^m on (397) and the one to follow here are worth considering because they are not terribly technical and they illustrate techniques that are conceptually important to path integrals.

Th^m/ Suppose \vec{F} is continuous on an open connected region D . If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D then \vec{F} is a conservative vector field on D ; that is \exists a function f such that $\vec{F} = \nabla f$

- In this course "connected" means path connected which means any two points in D can be connected by a path that is contained entirely in D . This is a topological concept.
- "Open" is also topological concept. D being open means that each point in D has a neighborhood about that point contained entirely in D .



$D = D_1 \cup D_2$
not connected,
the path from
P to Q
must go
outside D .
However, D_1 & D_2
are connected, but
they're not open.



open and connected
the ---- indicates
those points not
in S . Each $x \in S$
can have a nbhd
 N_x about x
entirely contained in S .

- Open, Closed, Bounded, Connected, Compact, Convergent, ... all of these ideas are treated seriously in advanced calculus then topology etc... We'll content ourselves with a few pictures.

Proof: Begin by picking $\vec{r}_0 = (a, b)$ a fixed point in D . We claim that the following is a potential function for \vec{F} ,

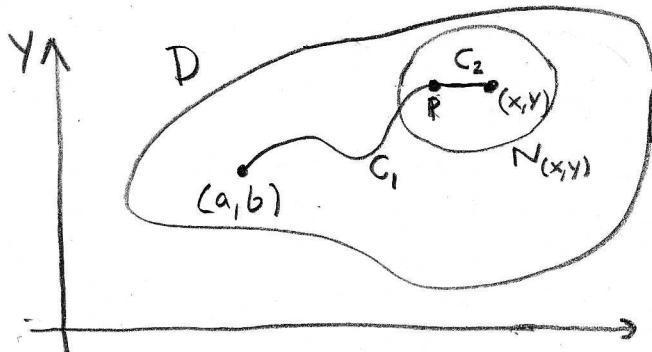
$$f(x, y) = \int_{(a, b)}^{(x, y)} \vec{F} \cdot d\vec{r}$$

Remark: In physics we define the electric potential V by

$$V(\vec{r}) = - \int_C \vec{E} \cdot d\vec{l} \rightarrow \vec{E} = -\nabla V$$

(the zero for the potential) $\rightarrow \Theta$ (we'll prove this shortly upto that)
minus sign that differs

Proof Continued: We claim $f(x, y) = \int_{(a, b)}^{(x, y)} \vec{F} \cdot d\vec{r}$. Let's draw a picture to start, notice because of path independence we didn't need to specify which path we take from (a, b) to (x, y) . Consider a particular choice



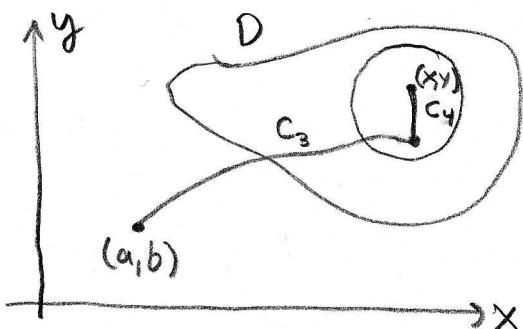
to $P = (x_0, y) \in N_{(x, y)}$ then C_2 goes horizontally ($dy = 0$) to (x, y) .

$$f(x, y) = \underbrace{\int_{(a, b)}^{(x_0, y)} \vec{F} \cdot d\vec{r}}_{\text{independent of } x} + \underbrace{\int_{C_2} \vec{F} \cdot d\vec{r}}_{\text{has } dy = 0}$$

Let $\vec{F} = \langle P, Q \rangle$,

$$\frac{\partial}{\partial x} [f(x, y)] = \frac{\partial}{\partial x} \left[\int_{C_2} P dx + Q dy \right] = \frac{\partial}{\partial x} \left[\int_{x_0}^x P(t, y) dt \right] = P(x, y).$$

On the other hand we could choose a vertical path inside $N_{(x, y)}$.



This time $C = C_3 \cup C_4$ and C_3 goes from (a, b) to (x, y_0) then C_4 goes from (x, y_0) to (x, y) along C_4 we have $dx = 0$.

$$f(x, y) = \underbrace{\int_{(a, b)}^{(x, y_0)} \vec{F} \cdot d\vec{r}}_{\text{independent of } y} + \underbrace{\int_{C_4} \vec{F} \cdot d\vec{r}}_{\text{has } dx = 0}$$

Let $\vec{F} = \langle P, Q \rangle$

$$\frac{\partial}{\partial y} [f(x, y)] = \frac{\partial}{\partial y} \left[\int_{C_4} P dx + Q dy \right] = \frac{\partial}{\partial y} \left[\int_{y_0}^y Q(x, t) dt \right] = Q(x, y).$$

Therefore, we find that $\nabla f = \langle P, Q \rangle = \vec{F}_{\parallel}$.

Remark: this proof easily generalizes to \mathbb{R}^n , we use same arguments of breaking up a path from (a_1, a_2, \dots, a_n) to (x_1, x_2, \dots, x_n) into pieces so that C_i doesn't depend on say x_k while C_j has $dx_i = 0$ for $i \neq k$. It's not hard to picture for a sphere $N(x, y, z)$ in \mathbb{R}^3 .