

## DIVERGENCE THEOREM

(421)

A simple solid region is one which can be encapsulated by a sphere, ellipsoid, cube etc... Its a bounded subset of  $\mathbb{R}^3$ , with no holes.

**Thm:** Let  $E$  be a simple solid region with  $S = \partial E$  the boundary given an outward orientation. Let  $\vec{F}$  be a vector field whose component functions have continuous partials on  $E$ . Then

$$\iint_{\partial E} \vec{F} \cdot d\vec{s} = \iiint_E (\nabla \cdot \vec{F}) dV$$

Proof: See Stewart §17.8, its geometry plus the FTC as usual.

**E172** Find flux  $\Phi_F = \iint_S \vec{F} \cdot d\vec{s}$  over the sphere  $S': x^2 + y^2 + z^2 = R^2$  where  $\vec{F} = \langle x, y, z \rangle$ . This example (Stolen from Stewart p.968) is tailor made for the divergence Thm. Notice

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_{\substack{x^2+y^2+z^2 \leq R^2 \\ E}} (\nabla \cdot \vec{F}) dV = 3 \iiint_E dV = 3 \cdot \frac{4}{3} \pi R^3 = 4\pi R^3 = \Phi_F$$

**E173** Let us consider the sphere of radius  $R$  again find flux of  $\vec{F} = \langle xy^2, yz^2, zx^2 \rangle$ . Note  $\nabla \cdot \vec{F} = y^2 + z^2 + x^2 = \rho^2$

$$\begin{aligned} \Phi_F &= \iint_S \vec{F} \cdot d\vec{s} = \iiint_E \rho^2 dV = \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \rho^2 \sin\phi d\rho d\phi d\theta \\ &= \int_0^R \rho^4 d\rho \int_0^\pi d\theta \int_0^\pi \sin\phi d\phi \\ &= \frac{1}{5} R^5 \cdot 2\pi \cdot 2 \\ &= \frac{4\pi R^5}{5} = \Phi_F \end{aligned}$$

**E174** Consider  $\vec{E} = \frac{kQ}{r^2} \hat{r}$  in the physics notation, so

$r^2 = x^2 + y^2 + z^2$  and  $0 \leq \phi \leq 2\pi$ ,  $0 \leq \theta \leq \pi$ ,  $s^2 = x^2 + y^2$ .

$$\begin{aligned}\nabla \cdot \left( \frac{kQ}{r^2} \hat{r} \right) &= \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 E_r] + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} [\sin \theta E_\theta] + \frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{kQ}{r^2} \right] \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} [kQ] \\ &= 0, \quad \therefore \quad \nabla \cdot \vec{E} = 0\end{aligned}$$

Then by the divergence theorem we calculate the flux of  $\vec{E}$  through the sphere  $x^2 + y^2 + z^2 = R^2$ ,

$$\iint_S \vec{E} \cdot d\vec{S} = \iiint (\nabla \cdot \vec{E}) dV = 0 \Rightarrow \Phi_E = 0 \quad (*)$$

Let's check this against explicit calculation of the surface integral, note  $dS = R^2 \sin \theta d\theta d\phi \hat{r}$  thus

$$\begin{aligned}\iint_S \vec{E} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^\pi \frac{kQ}{R^2} \cdot R^2 \sin \theta d\theta d\phi \quad \text{recall } k = \frac{1}{4\pi\epsilon_0} \\ &= kQ \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = 4\pi kQ = \frac{4\pi}{4\pi\epsilon_0} Q = \frac{Q}{\epsilon_0}.\end{aligned}$$

So which is it  $\Phi_E = 0$  or  $\Phi_E = Q/\epsilon_0$ ?

RESOLUTION TO PARADOX:  $\nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = 4\pi \delta(\vec{r})$  where  $\delta(\vec{r})$

is the 3-d Dirac Delta function,  $\int f(\vec{r}) \delta(\vec{r}) dV = f(0)$ . In simple terms  $\delta(\vec{r}) = 0$  everywhere except at  $\vec{r} = 0$  where it is infinite. Then  $\nabla \cdot \vec{E} = \frac{Q}{\epsilon_0} \delta(\vec{r})$ , then one of Maxwell's Eq's is  $\nabla \cdot \vec{E} = \rho/\epsilon_0$  thus for a point charge  $Q$  centered at the origin the charge density is  $\rho(\vec{r}) = Q \delta(\vec{r})$ . Mathematically our sol'n was bogus since  $\text{dom}(\vec{E}) \not\ni (0,0,0)$ . It had a hole. You can look at p. 1138-1139 to see how Stewart dodges this.

Remark: You'll see  $\delta(x)$  in ma 341 when you study discontinuous forcing functions on springs and things. The  $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$  where  $\int f(x)\delta(x)dx = f(0)$ . These Dirac Delta functions turn integration into evaluation. The mathematics to seriously do these things wasn't known until the early 20<sup>th</sup> century, see the work of Schwartz. If you object to point charges you could insist that the charge  $Q$  was smeared out over some tiny sphere that would give a density of:

$$\rho_1 = \begin{cases} \frac{Q}{\frac{4}{3}\pi a^3} \frac{4}{3}\pi r^3 & 0 \leq r \leq a \\ 0 & r > a \end{cases}$$



$$\rho_1 = Q \delta(\vec{r}) \quad (\text{this picture is way too big, it's a point}) \cdot Q$$

The interesting thing is that for  $r > a$  both  $\rho_1$  &  $\rho_2$  yield the same field. This is Gauss Law,

$$\frac{Q_{\text{enc}}}{\epsilon_0} = \iint_S \vec{E} \cdot d\vec{A} = \iiint_S (\nabla \cdot \vec{E}) dV = \iiint_V \rho_1 / \epsilon_0 dV = \iiint_V \rho_2 / \epsilon_0 dV$$

↓ divergence Th<sup>m</sup>      ↓  $\nabla \cdot \vec{E} = \rho / \epsilon_0$       Gauss' Law in differential form

By symmetry  $\iint \vec{E} \cdot d\vec{A} = 4\pi R^2 / |\vec{E}|$  and  $\vec{E} = E_r \hat{r}$ ,  $E_\theta = E_\phi = 0$ . thus

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \hat{r}}$$

Remark: the Divergence Th<sup>m</sup> is also called Gauss Th<sup>g</sup>. The calculation sketched above connects the so-called integral & differential formulations of Gauss' Law

$$\oint_E \frac{d\vec{A}}{\epsilon_0} = \frac{Q}{\epsilon_0} \Leftrightarrow \nabla \cdot \vec{E} = \rho / \epsilon_0$$