

MULTIVARIATE INTEGRATION

To begin we will study double & triple integrals over box-like or rectangular regions, these are as easy as integrations from calc. I & II. Then we find how to integrate over type I & II regions in the xy -plane and general volumes in xyz -space, this is not as easy but once you understand the importance of graphing and set-up it becomes clear. After exhausting topics in Cartesian Coordinates we'll study the Jacobian, this will give us a derivation of how to integrate in sphericals or cylindricals or polars or whatever system of coordinates you might invent. Then we apply our Jacobian theory to do integrals in polars, cylindricals and sphericals. We scatter select applications throughout our discussion.

Definitions: integrals are defined as the limit of a weighted sum over f ,

$$\int_a^b f(x) dx \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \quad : \quad \Delta x = \frac{b-a}{n}$$

$$\iint_R f(x, y) dA \equiv \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^k f(x_i^*, y_j^*) \Delta x \Delta y$$

$$R = [a, b] \times [c, d] \quad \text{and} \quad \Delta x = \frac{b-a}{n} \quad \text{while} \quad \Delta y = \frac{d-c}{k}$$

$$\iiint_B f(x, y, z) dV \equiv \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{l=1}^k f(x_i^*, y_j^*, z_l^*) \Delta x \Delta y \Delta z$$

$$B = [a, b] \times [c, d] \times [p, q] \quad \text{and} \quad \Delta x = \frac{b-a}{n}, \quad \Delta y = \frac{d-c}{n}, \quad \Delta z = \frac{q-p}{k}$$

We note that in Cartesian coordinates $dA = dx dy = \text{infinitesimal area element in } xy\text{-plane}$. $dV = dx dy dz = \text{infinitesimal volume element}$. As in calc I and II, the sample points are chosen randomly, but it doesn't matter in the limit. In practice the limit is rarely seen, instead the F.T.C. or evaluation rule and here the Fubini Th^m will keep us from ever using the defⁿ directly (THANKFULLY!)

Several Properties of the integral follow directly from the properties of the limit itself: Let R be a rectangular region

$$\iint_R [f(x,y) + g(x,y)] dA = \iint_R f(x,y) dA + \iint_R g(x,y) dA$$

$$\iint_R c f(x,y) dA = c \iint_R f(x,y) dA$$

$$f(x,y) \geq g(x,y) \quad \forall (x,y) \in R \Rightarrow \iint_R f(x,y) dA \geq \iint_R g(x,y) dA.$$

Likewise for $f(x,y,z)$ and $g(x,y,z)$ over a box-like region. We assume that f, g are continuous most everywhere. Meaning we can integrate $f(x,y)$ if it has a finite number of curve discontinuities, or $f(x,y,z)$ if it has a finite number of planar discontinuities. We just chop the integral into a finite # of regions on which f is continuous.

FUBINI'S THM (WEAK FORM): known to Cauchy for continuous f in early 19th century.
Let $R = [a,b] \times [c,d]$ and let f be a mostly continuous funct. of (x,y)

$$\iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$$

where the expressions on the RHS are "iterated integrals" which you work inside out, treating the outside variable as a constant to begin.

E87 Let $R = [0,\pi] \times [0,2]$ that is $0 \leq x \leq \pi$ and $0 \leq y \leq 2$. Integrate $f(x,y) = \sin(x) + y$ over R .

$$\iint_R f(x,y) dA = \int_0^\pi \left(\int_0^2 [\sin(x) + y] dy \right) dx : \text{parentheses added to emphasize the order of operations here.}$$

$$= \int_0^\pi \left[y \sin(x) + \frac{1}{2} y^2 \right]_0^2 dx : \text{note } \sin(x) \text{ is regarded as a constant in the } dy \text{ integration.}$$

$$= \int_0^\pi [2 \sin(x) + 2] dx$$

$$= -2 \cos(x) \Big|_0^\pi + 2x \Big|_0^\pi$$

$$= -2 \cos \pi + 2 \cos(0) + 2\pi$$

$$= 4 + 2\pi$$

Exercise: compute $\iint_{[0,2]\times[0,\pi]} (\sin(x) + y) dx dy$
you should get same answer.

E88 Let $R = \{(x, y) \mid 0 \leq x \leq \pi/2, 0 \leq y \leq 2\}$.

$$\begin{aligned} \iint_R y \cos(xy) dA &= \int_0^2 \left(\int_0^{\pi/2} y \cos(xy) dx \right) dy : \int \cos(ax) dx = \frac{1}{a} \sin(ax) + C \\ &= \int_0^2 \left(\frac{y}{\pi} \sin(xy) \Big|_0^{\pi/2} \right) dy \\ &= \int_0^2 \left(\sin\left(\frac{\pi y}{2}\right) - \sin(0) \right) dy \\ &= -\frac{2}{\pi} \cos\left(\frac{\pi y}{2}\right) \Big|_0^2 = -\frac{2}{\pi} (\cos(\pi) - \cos(0)) = \boxed{\frac{4}{\pi}} \end{aligned}$$

Remark: the order of S here is easier than if we reversed to $dy dx$

GEOMETRY: the double integral of $f(x, y)$ over R is the volume of the solid bounded by $z = f(x, y)$, $z = 0$ and $(x, y) \in R$. The Thm of Fubini can be seen as merely saying you can slice up the volume along x or y crosssections. Infinitesimally

$$dV = \underbrace{(z_{\text{top}} - z_{\text{bottom}})}_{\substack{\text{height of} \\ \text{the box}}} \underbrace{dx dy}_{\substack{\text{area of} \\ \text{the box}}}$$

So if $z_{\text{bottom}} = 0$ and $z_{\text{top}} \geq 0$ then we get the volume, however as in $\int f(x) dx$ we count volume below the xy -plane as negative so the integral calculates the "signed" volume. I'm not even going to pretend I can draw these things... I'll let Maple do the artistry. I do hope we can all see these things in our "minds eye" in the end.

E89 Let $B = \{(x, y, z) \mid 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\}$

$$\begin{aligned} \iiint_B dV &= \int_0^c \int_0^b \int_0^a dx dy dz \\ &= \int_0^c \int_0^b x \Big|_0^a dy dz \\ &= \int_0^c \left(\int_0^b a dy \right) dz \\ &= \int_0^c ab dz \\ &= \boxed{abc = V} \end{aligned}$$

- If we integrate 1 over B we find the volume of B . Likewise if we had integrated 1 over a rectangle $R \subset \mathbb{R}^2$ we would have found the area.

E90 Let $B = [0, 1] \times [0, 2] \times [0, 3]$. Let $\rho = \frac{dm}{dV} = xyz$. Consider,

$$\iiint_B xyz \, dV = \int_0^3 \int_0^2 \int_0^1 xyz \, dx \, dy \, dz$$

$$= \int_0^3 z \, dz \int_0^2 y \, dy \int_0^1 x \, dx \quad : \text{only allowed if our functions factors into functions of } x, y, z$$

$$= \frac{1}{2} z^2 \Big|_0^3 \cdot \frac{1}{2} y^2 \Big|_0^2 \cdot \frac{1}{2} x^2 \Big|_0^1 \\ f(x, y, z) = f_1(x)f_2(y)f_3(z)$$

$$= \frac{1}{8} (3)^2 (2)^2$$

$$= \boxed{\frac{27}{2}}$$

What is the meaning of such an integration? Well if $\rho = \text{mass density} = dm/dV$ then $\rho dV = dm$ thus

$$m = \int_B dm = \iiint_B \rho \, dV = \boxed{\frac{27}{2}} = \text{mass of object } B \text{ with density } \rho = xyz.$$

Or you could interpret it as $\rho = dq/dV = \text{charge/volume}$
 so $q = \int_B \rho \, dV = \frac{27}{2} = \text{charge of object } B$. I'm
 sure you could imagine other densities.

E91 Another interpretation of $\iint_R f(x, y) \, dA$ is that $f(x, y)$ represents an area density. So say $f(x, y) = \sigma(x, y)$

$$\sigma(x, y) = \frac{dq}{dA} \Rightarrow q = \iint_R \sigma(x, y) \, dA = \text{charge on the planar region } R$$

$$\sigma(x, y) = \frac{dm}{dA} \Rightarrow m = \iint_R \sigma(x, y) \, dA = \text{mass of the rectangle } R.$$

E92 Another interpretation of $\int_a^b f(x) \, dx$ is that $f(x)$ represents a linear density. So say $f(x) = \lambda(x)$ and,

$$\lambda(x) = \frac{dq}{dx} \Rightarrow q = \int_a^b \lambda(x) \, dx, \quad \lambda = \frac{dm}{dx} \Rightarrow m = \int_a^b \lambda(x) \, dx$$

Remark: linear density is more exciting once we know about line-integrals (which are actually along curves generally). Note "density" is multifaceted.