

KEPLER'S LAWS OF PLANETARY MOTION

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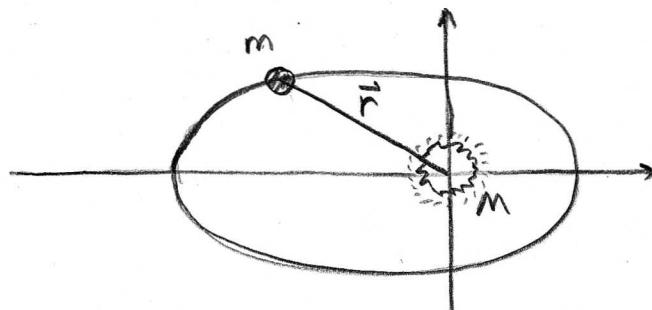
In antiquity there have been radically different views of the universe at large and the motion or lack of motion of the earth through it. At the time of Kepler the heliocentric view of Copernicus (1473-1543) had taken hold, but astronomers insisted that planets traveled in circles, then circles on top of circles on top of circles... This system of "perfect" circles were known as epicycles. Epicycles worked quite well but Kepler (1571-1630) found them unnatural. Kepler instead thought he could explain the motion of planets by a few simple rules. He found these rules empirically by studying the exquisite data taken by Tycho Brahe. These laws were chosen simply to fit the data. Only later were these laws derived from basic physical law. By the way, much of modern physics are still like Kepler's Laws, it is always the dream/goal/aspiration to derive known phenomenological law from basic principles. There is some controversy as to who first derived Kepler's Laws, many credit Newton himself others credit Johann Bernoulli in 1710. The incredible thing is that we can derive the laws in a few short pages. Our notation and understanding of vector calculus is several hundred years in advance, so ordinary folks like myself can grasp the proof.

Set-up

Kepler's laws for the Sun and a single planet are:

- 1.) The orbit of the planet is elliptical with the sun at a focus.
- 2.) During equal times the planet sweeps out equal areas in the ellipse.
- 3.) $T^2 \propto a^3$ where $T = \text{period of planet's orbit}$, $a = \text{length of semimajor axis of ellipse}$.

We place the origin at the sun. We expect that



• My proof of Kepler's Laws follows Colley's of §3.1 fairly closely.

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Proposition: The motion of the planet lies in a plane which also contains the sun if we assume Newton's Universal Law of Gravitation governs the motion through Newton's Laws.

Proof: our goal is to show that $\vec{r} \times \vec{v} = \vec{C}$ for some constant vector \vec{C} . This will show that planet moves in a plane with normal \vec{C} . Note,

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}}_{\vec{v} \times \vec{v} = 0} = \vec{r} \times \vec{a}.$$

Recall in our current notation that $\vec{r} = r\hat{r}$ and Newton tells us that,

$$\vec{F} = m\vec{a} = -\frac{GmM}{r^2}\hat{r} = -\frac{GmM}{r^3}\vec{r}$$

$$\therefore \vec{a} = -\frac{GM}{r^3}\vec{r} \quad \text{thus } \vec{a} \parallel \vec{r}$$

m = mass of planet

M = mass of sun

G = Gravitational Constant.

$$\Rightarrow \vec{a} \times \vec{r} = 0 \Rightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r} \times \vec{a} = 0 \therefore \vec{r} \times \vec{v} = \vec{C}$$

Theorem / Kepler's 1st. Law: The planet's orbit is an ellipse with sun at one focus

Proof: this will take a little work so be patient, lets get a better hold on \vec{C} ,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

Apply this to the following,

$$\vec{C} = \vec{r} \times \vec{v} = r\hat{r} \times [\dot{r}\hat{r} + r\frac{d\hat{r}}{dt}] = \underbrace{r^2 \hat{r} \times \frac{d\hat{r}}{dt}}_{\text{I.}} = \vec{C}$$

Calculate then, using (I)

$$\vec{a} \times \vec{C} = \left(-\frac{GM}{r^2}\hat{r}\right) \times \left(r^2 \hat{r} \times \frac{d\hat{r}}{dt}\right)$$

$$= -GM[\hat{r} \times (\hat{r} \times \frac{d\hat{r}}{dt})]$$

see §9.4 #30

$$= GM[(\hat{r} \times \frac{d\hat{r}}{dt}) \times \hat{r}] : \text{recall } A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

$$= GM[(\hat{r} \cdot \hat{r})\frac{d\hat{r}}{dt} - (\hat{r} \cdot \frac{d\hat{r}}{dt})\hat{r}] : \hat{r} \cdot \hat{r} = 1 \Rightarrow 2\hat{r} \cdot \hat{r} = 0 \Rightarrow \hat{r} \cdot \hat{r} = 0,$$

$$= GM \frac{d\hat{r}}{dt}$$

$$= \underbrace{\frac{d}{dt}(GM\hat{r})}_{\text{II.}} = \vec{a} \times \vec{C}$$

Proof of Kepler's 1st Law continued

We may derive another identity for $\vec{a} \times \vec{c}$,

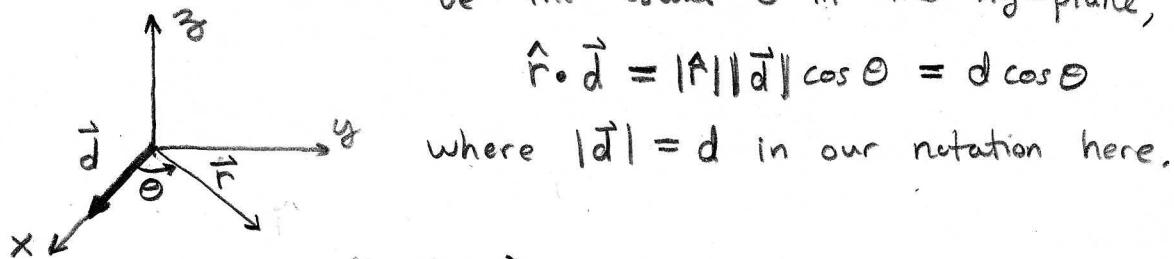
$$\vec{a} \times \vec{c} = \frac{d\vec{v}}{dt} \times \vec{c} + \vec{v} \times \frac{d\vec{c}}{dt} : \text{added zero since } \frac{d\vec{c}}{dt} = 0.$$

$$= \underbrace{\frac{d}{dt} [\vec{v} \times \vec{c}]}_{\text{III}} : \text{using identity (V.) on 265}$$

Thus comparing II & III we find

$$\frac{d}{dt}(GM\hat{r}) = \frac{d}{dt}(\vec{v} \times \vec{c}) \therefore \underbrace{\vec{v} \times \vec{c}}_{\text{IV}} = GM\hat{r} + \vec{d}$$

where \vec{d} is a constant vector, it lies in the orbital plane since $\vec{v} \times \vec{c}$ and \hat{r} do. Now choose coordinates in the orbital plane so that \vec{d} lines up with the x -axis. Let θ be the usual θ in the xy -plane,



$$\hat{r} \cdot \vec{d} = |\hat{r}| |\vec{d}| \cos \theta = d \cos \theta$$

where $|\vec{d}| = d$ in our notation here.

Now consider the length of \vec{c} squared,

$$\begin{aligned} c^2 &= \vec{c} \cdot \vec{c} \\ &= (\vec{r} \times \vec{v}) \cdot \vec{c} \\ &= \vec{r} \cdot (\vec{v} \times \vec{c}) : \text{using identity (V.) of 248} \\ &= r\hat{r} \cdot [GM\hat{r} + \vec{d}] : \text{using IV. we found just above.} \\ &= GMr + r\hat{r} \cdot \vec{d} \\ &= GMr + rd \cos \theta \\ &= r(GM + d \cos \theta) \end{aligned}$$

Therefore we solve for $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2}$ (we're in $z=0$) and obtain the eq² of an ellipse (or parabola or hyperbola)

$$r = \frac{c^2}{GM + d \cos \theta} = \frac{c^2/GM}{1 + (d/GM) \cos \theta} = \boxed{\frac{P}{1 + e \cos \theta}} = r$$

where we define $P = c^2/GM$ and the eccentricity $e = d/GM$. This is an ellipse in polar coordinates. Since you've likely not seen that recently (or maybe never) we'll connect to

Proof of Kepler's 1st Law continued

the usual Cartesian eqn's for the ellipse. The details will be of use to us in proving the 3rd Law of Kepler later on.

$$r = \frac{P}{1+e\cos\theta} \Rightarrow r = P - e\cos\theta$$

Trying to convert the polar coordinates (r, θ) to (x, y) where $x = r\cos\theta$ and $y = r\sin\theta$. We see, using $x = r\cos\theta$

$$r = P - ex$$

$$r^2 = x^2 + y^2 = P^2 - 2epx + e^2x^2$$

$$x^2 - e^2x^2 + y^2 + 2epx = P^2$$

$$x^2(1-e^2) + 2epx + y^2 = P^2$$

$$x^2 + \frac{\partial ep}{1-e^2}x + \frac{y^2}{1-e^2} = \frac{P^2}{1-e^2} \quad : \quad \underline{\text{assume } e \neq \pm 1}$$

$$\left(x - \frac{ep}{1-e^2}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{P^2}{1-e^2} + \frac{e^2p^2}{(1-e^2)^2} = \frac{P^2 - e^2p^2 + e^2p^2}{(1-e^2)^2} = \frac{P^2}{(1-e^2)^2}$$

$$\therefore \boxed{\frac{\left(x - \frac{ep}{1-e^2}\right)^2}{P^2/(1-e^2)^2} + \frac{y^2}{P^2/(1-e^2)} = 1} \quad \begin{array}{l} \text{ellipse or hyperbola} \\ (0 < e < 1) \quad (e > 1) \end{array}$$

This is an ellipse with center $(ep/(1-e^2), 0)$ and it has semi major axis length $a = P/(1-e^2)$ and semiminor axis $b = P/\sqrt{1-e^2}$.

Remark: recall that we defined $P = c^2/GM$ so $P > 0$ and we need not worry about \sqrt{P} by P . Now $e = d/GM > 0$ so we can rule out $e = -1$ as a problem. Notice we have division by $\sqrt{1-e^2}$ as part of our sol^{1/2}, this only makes sense if $0 < e < 1$. The case $e = 1$ needs separate treatment. Motion in the case $0 < e < 1$ is that of planets.

$$\underline{e = 1} \quad r = P - r\cos\theta \quad \therefore r^2 = (P-x)^2 = P^2 - 2xP + x^2$$

$$\text{that is } x^2 + y^2 = P^2 - 2xP + x^2 \Rightarrow 2xP = P^2 - y^2$$

$$\therefore \boxed{x = P/2 - y^2/2P} \quad \text{parabola}$$

Remark: One nice resource for background on conic-sections and polar coordinates is "Precalculus, Concepts through functions" Sullivan & Sullivan. There is just about all the cases you can imagine, rotated ellipses for example.

Thⁿ/ KEPLER'S 2nd LAW: During equal times a planet sweeps through equal areas.

Proof: Pick a point P_0 at angle Θ_0 . The later in this course we will learn that the area in polar coordinates swept by the region from Θ_0 to Θ is simply

$$A(\Theta) = \int_{\Theta_0}^{\Theta} \frac{1}{2} r^2 d\beta$$

We seek to show that $\frac{dA}{dt} = \text{constant}$. Consider then

$$\frac{dA}{d\Theta} = \frac{d}{d\Theta} \int_{\Theta_0}^{\Theta} \frac{1}{2} r^2 d\beta = \frac{1}{2} r^2 \quad \text{by F.T.C.}$$

Then the chain rule tells us

$$\frac{dA}{dt} = \frac{dA}{d\Theta} \frac{d\Theta}{dt} = \frac{1}{2} r^2 \frac{d\Theta}{dt}$$

Notice that $\hat{r} = \langle \cos \Theta, \sin \Theta \rangle$ thus diff. implicitly, remember $\Theta = \Theta(t)$.

$$\frac{d\hat{r}}{dt} = \langle -\sin \Theta, \cos \Theta \rangle \frac{d\Theta}{dt} = \langle -\sin \Theta, \cos \Theta, 0 \rangle \frac{d\Theta}{dt} \quad (\text{we've been suppressing the } 3\text{-comp.})$$

$$\textcircled{B}, \textcircled{I} \Rightarrow \vec{c} = r^2 \left(\hat{r} \times \frac{d\hat{r}}{dt} \right) = r^2 \frac{d\Theta}{dt} \langle \cos \Theta, \sin \Theta, 0 \rangle \times \langle -\sin \Theta, \cos \Theta, 0 \rangle$$

$$\vec{c} = r^2 \frac{d\Theta}{dt} \langle 0, 0, 1 \rangle \quad \therefore c = r^2 \frac{d\Theta}{dt}$$

Hence $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\Theta}{dt} = \frac{c}{2} = \text{constant.} //$

Thⁿ/ KEPLER'S 3rd Law: $T^2 = K a^3$ where T is the orbital period and a is the length of the semimajor axis, $K = \text{some constant}$

Proof: I proved back on pg. 138 in E7 that the area of an ellipse is $A = \pi ab$. On the other hand we could say that $dA = \frac{dA}{dt} dt$ and integrate over a whole orbit to find

$$\pi ab = \int_0^T \frac{dA}{dt} dt = \int_0^T \frac{c}{2} dt = \frac{cT}{2} \quad \therefore T = \frac{2\pi ab}{c} \quad \therefore T^2 = \frac{4\pi^2 a^2 b^2}{c^2}$$

notice that $a^2 = p^2/(1-e^2)^2$ and $b^2 = p^2/(1-e^2)$, also $c^2 = GMp$.

$$T^2 = \frac{4\pi^2}{GMp} \frac{p^2}{(1-e^2)^2} \cdot \frac{p^2}{(1-e^2)} = \frac{4\pi^2}{GM} \left(\frac{p}{1-e^2} \right)^3 = \boxed{\frac{4\pi^2 a^3}{GM} = T^2} //$$

It is interesting that $K = \frac{4\pi^2}{GM}$ is independent of the planets mass. all the planets orbit under the same K -value.

Remark: I have spent some effort presenting Kepler's Laws.
I may ask you to prove some subset, I'll pin it down
in the test review sheet.

Remark: There is another method of proving Kepler's Laws
that begins with the two-body Lagrangian for a central
potential (well force really but $\vec{F} = f(r)\hat{r} \Rightarrow U = U(r) \dots$). In
that derivation one need not assume the sun is at the
origin. Instead you consider the center of mass to be
at the origin and work out how the reduced mass
 μ orbits. Anyway its very beautiful, take Mechanics at the Junior/Senior
level to see the more general derivation. Also they
will actually find $\vec{r}(t)$ explicitly as opposed to the
indirect arguments we have offered (or rather stolen from Colley \textcircled{C}).