

LAGRANGE MULTIPLIERS

Our goal is to find the extrema of $f(x, y, z)$ subject to a constraint condition $g(x, y, z) = 0$. If f has an extreme value at $P = (x_0, y_0, z_0)$ then the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ composed with f will have an extreme value at t_0 , where $\vec{r}(t_0) = (x_0, y_0, z_0)$ thus

$$0 = \frac{d}{dt} \left[f(x(t), y(t), z(t)) \right] \Big|_{t_0} = \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) \Big|_{t_0}$$

$$= (\nabla f)(P) \cdot \vec{r}'(t_0) = 0$$

On the other hand this curve can also be composed with g

$$\frac{d}{dt} [g(x(t), y(t), z(t))] = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} + \frac{\partial g}{\partial z} \frac{dz}{dt} = (\nabla g) \cdot \vec{r}'(t) = \frac{d}{dt}(k) = 0,$$

where we get zero since $g(x, y, z) = k$, thus we also have $(\nabla g)(P) \cdot \vec{r}'(t_0) = 0$.

Therefore we deduce that $\nabla f = \lambda \nabla g$ at the extremum.

METHOD OF LAGRANGE MULTIPLIERS:

Assuming that $f(x, y, z)$ has max/min values on the surface $g(x, y, z) = k$ with $\nabla g \neq 0$ we can find them as follows

(i) set $\nabla f = \lambda \nabla g$ and use $g(x, y, z) = k$ to simplify and find sol's.

(ii) evaluate the sol's from (i) to see where f attains its min/max

If $f = f(x, y)$ and $g = g(x, y) = k$ then do the same, just with 2-dim'l gradients.

[E84] Let $f(x, y) = xy$ find extrema of f on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Identify that $g(x, y) = x^2/8 + y^2/2 = 1$. Consider then

$$\nabla f = \lambda \nabla g \Rightarrow \langle y, x \rangle = \lambda \langle x/4, y \rangle$$

$$\Rightarrow 4y = 2x \text{ and } x = 2y$$

$$\Rightarrow y = \frac{2}{4} (2y)$$

$$\Rightarrow y(1 - 2^2/4) = 0$$

$$\Rightarrow y = 0 \text{ or } \lambda = \pm 2.$$

E84 continued We've gathered that $y=0$ or $x=\pm 2$ yield the extrema,

$y=0$ then $x=2y$ and so $x=0$ as well but $(0,0)$ not on ellipse.

$x=2$ $x=2y$ and $4y=2x$ a.k.a. $y=\frac{1}{2}x$

$$\frac{x^2}{8} + \frac{y^2}{2} = \frac{x^2}{8} + \frac{1}{2} \cdot \frac{x^2}{4} = \frac{1}{4}x^2 = 1 \therefore x^2 = 4$$

$\therefore x = \pm 2$ and $y = \frac{1}{2}(\pm x) = \pm 1$ so $(-2, -1)$ or $(2, 1)$

$f(-2, -1) = (-2)(-1) = 2$ while $f(2, 1) = 2(1) = 2$.

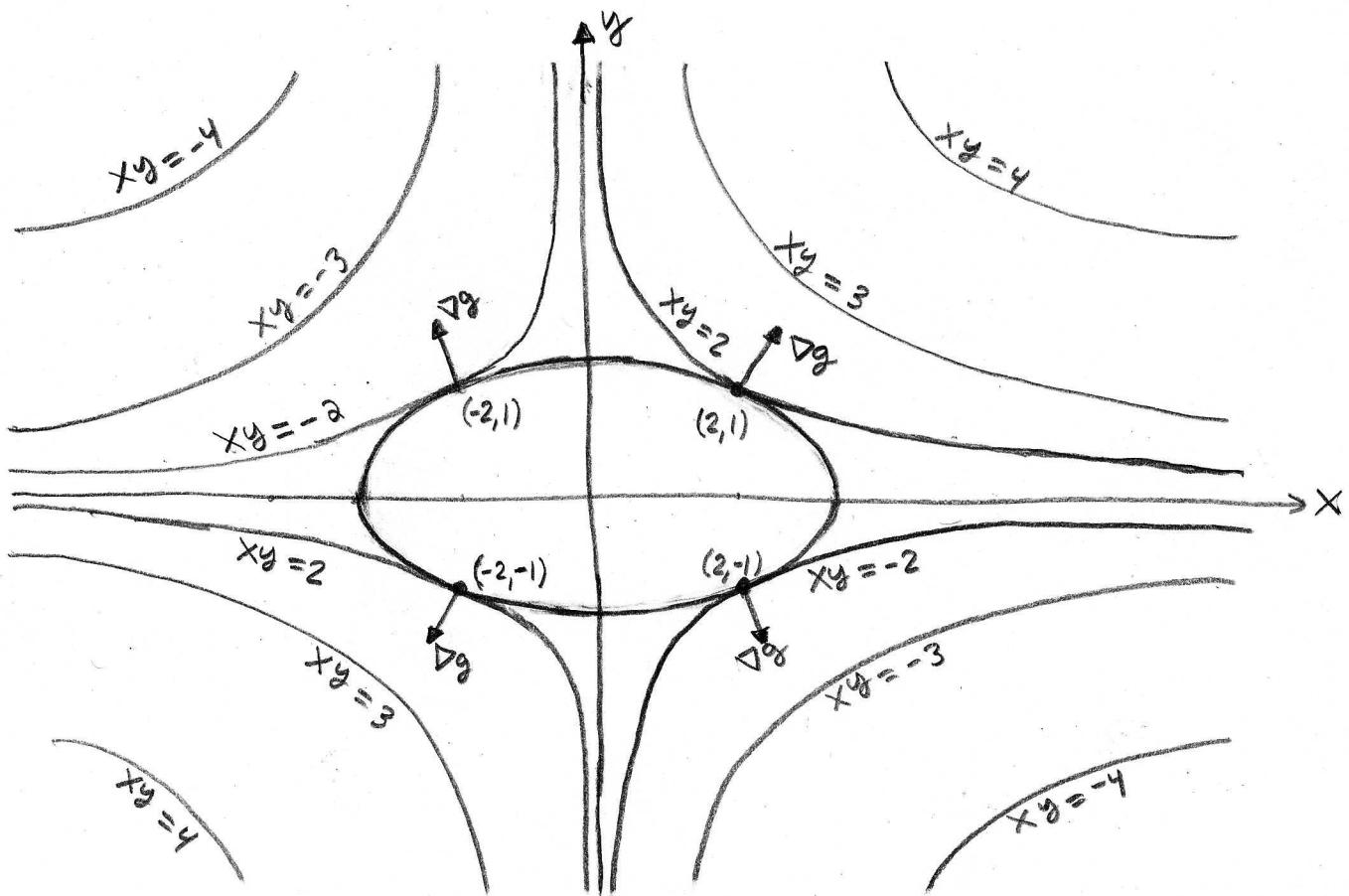
$x=-2$ $x=-2y$ and $4y=-2x$ that is $y = -\frac{1}{2}x$

$$\frac{x^2}{8} + \frac{y^2}{2} = \frac{x^2}{8} + \frac{1}{2} \cdot \frac{x^2}{4} = \frac{1}{4}x^2 = 1 \therefore x^2 = 4$$

$\therefore x = \pm 2$ and $y = \frac{1}{2}(\mp x) = \mp 1$ so $(-2, 1)$ or $(2, -1)$

$f(-2, 1) = -2$ while $f(2, -1) = -2$.

The extreme values are 2 and -2. The max is 2 which is reached at $(-2, -1)$ and $(2, 2)$ while the min. is obtained at $(-2, 1)$ and $(2, -1)$.



You can appreciate from the geometry why Lagrange's Method worked here.

E85 Find the point on the plane $z = x + y$ that is closest to the point $(1, 1, 0)$. In other words minimize $f(x, y, z) = (x-1)^2 + (y-1)^2 + z^2$ subject to $g(x, y, z) = x + y - z = 0$.

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-1), 2(y-1), 2z \rangle = \lambda \langle 1, 1, -1 \rangle$$

$$\Rightarrow \begin{cases} 2(x-1) = \lambda \\ 2(y-1) = \lambda \\ 2z = -\lambda \end{cases}$$

$$\Rightarrow \lambda/2 = x-1 = y-1 = -z$$

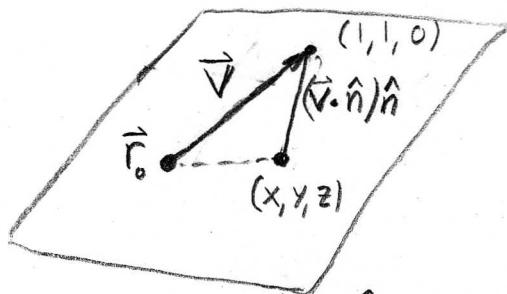
$$\Rightarrow x = y \text{ and } z = 1 - y$$

$$\Rightarrow \text{since } z = x + y = 2y = 1 - y$$

$$\therefore 3y = 1 \text{ thus } y = 1/3 = x \text{ and } z = 1 - 1/3 = 2/3.$$

the closest point on the plane $z = x + y$ to the point $(1, 1, 0)$ is $(1/3, 1/3, 2/3)$

Lets check our answer geometrically:



$$\vec{v} = (1, 1, 0) - \vec{r}_0$$

$$(\vec{v} \cdot \hat{n}) \hat{n} = \text{proj}_{\hat{n}}(\vec{v})$$

$$(x, y, z) = (1, 1, 0) - (\vec{v} \cdot \hat{n}) \hat{n}.$$

We just need to find a normal of the plane and a point on the plane. Choose $\vec{n} = \langle 1, 1, -1 \rangle$ so that $\hat{n} = \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle$ and $\vec{r}_0 = \langle 0, 0, 0 \rangle$. Hence,

$$\vec{v} = (1, 1, 0)$$

$$\text{proj}_{\hat{n}}(\vec{v}) = \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle \cdot \langle 1, 1, 0 \rangle \hat{n} = \frac{2}{\sqrt{3}} \hat{n} = \frac{2}{3} \langle 1, 1, -1 \rangle.$$

$$(1, 1, 0) - \text{proj}_{\hat{n}}(\vec{v}) = (1, 1, 0) - \frac{2}{3} (1, 1, -1) = \boxed{\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3} \right)}$$

this was our intuitive solⁿ we used in the early portion of this course, the closest point falls on the normal line connecting the point and the plane.

E86 A rectangular box without a lid is made from 12 square units of material. Find the maximum volume of such a box. That is maximize $V = xyz$ subject to $g = 2xz + 2yz + xy = 12$.

$$\nabla V = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2z+y, 2z+x, 2x+2y \rangle$$

$$yz = \lambda(2z+y) \Rightarrow xyz = \lambda(2zx+yz)$$

$$xz = \lambda(2z+x) \Rightarrow xyz = \lambda(2zy+xy)$$

$$xy = \lambda(2x+2y) \Rightarrow xyz = \underbrace{\lambda(2xz+2yz)}_{\text{using } g=0} = \lambda(12-xy)$$

Thus $\lambda(2zx+yz) = \lambda(2zy+xy) = \lambda(12-xy)$. We can divide by λ since $\lambda=0 \Rightarrow xyz=0$. Note then

$$2zx+yz = 2zy+xy = 12-xy$$

$$\Rightarrow 2zx = 2zy \Rightarrow x=y \quad (\text{note } z=0 \text{ is not a useful value.})$$

$$\text{Next notice } 2zy+xy = 2xz+2yz \Rightarrow y^2 = 2yz \therefore y = 2z.$$

$$\text{Then } 2xz + 2yz + xy = 4z^2 + 4z^2 + 4z^2 = 12z^2 = 12.$$

Hence $z = \pm 1$. Our material only comes in positive lengths

So $z = 1$ hence $x = y = 2$. The box is $2 \times 2 \times 1$.

Remark: there are a couple examples in your text I haven't stolen.

Remark: In the study of Lagrangian Mechanics if one has the constraint $g=0$ then it can be implemented by adding λg to the Lagrangian. $L \rightarrow L + \lambda g$. Then one finds the eq's of motion and at the end puts $g=0$. In that theory the Lagrange multipliers λ are found to be the forces needed to maintain the constraint eq's on the motion of the physical body. It is one of the most beautiful chapters in Classical Mechanics, take a course to see this in some detail, or buy an old copy of Goldstein's Classical Mechanics, its like \$10, its a classic. I sketch the method on the next pg (not req'd topic)

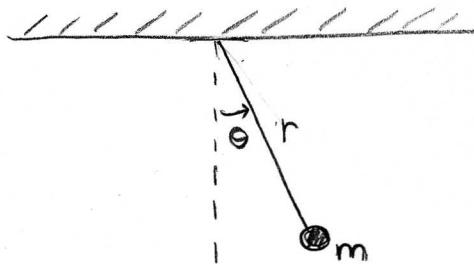
Digression: Lagrange Multipliers in Classical Mechanics:

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You may ignore this if you wish. The Lagrangian in classical mechanics is the function $L = T - U$ where $T = \text{kinetic energy}$ and $U = \text{potential energy}$. The eq's of motions follow from minimizing the action (Hamilton's Principle) and are called the Euler-Lagrange Eq's.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \leftarrow \begin{array}{l} \text{replaces Newton's Law} \\ \text{conceptually, energy} \\ \text{not force is primary} \end{array}$$

To impose constraints one may add Lagrange Multipliers to encode those constraints. I'll illustrate with the simple pendulum, the constraint is $r = l$ a.k.a. $f = r - l = 0$.



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = -mgr \cos \theta$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr \cos \theta$$

Then the technique of Lagrange Multipliers adds the term on the RHS

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \lambda \frac{df}{dr} \Rightarrow m\ddot{r} - mr\dot{\theta}^2 - mg \cos \theta = \lambda \quad \leftarrow \text{Lagrange multiplier}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) + mgr \sin \theta = 0$$

Then we have the following eq's to solve, subject to $r = l$

$$m\ddot{r} - mr\dot{\theta}^2 - mg \cos \theta = \lambda$$

$$\frac{d}{dt}(mr^2\dot{\theta}) + mgr \sin \theta = 0$$

This example is kinda silly since the constraint is so trivial, once we apply $r = l$ we find $\lambda = -ml\dot{\theta}^2 - mg \cos \theta$ and the eq's

$$\frac{d}{dt}(ml^2\dot{\theta}^2) + mgl \sin \theta = 0$$

These can be solved for small θ where $\sin \theta \approx \theta$. It can be seen that λ is the force that enforces the constraint $r = l$. Generally Lagrange multipliers allow us to solve the

eq's of motion subject to some geometric (even time dependent) constraint w/o knowing the forces that cause the motion to be constrained. Neat thing is the method shows what the forces are.

