Integral Vector Calculus: Overview

We have studied how to differentiate vector fields. We saw that three main operations were of interest. The gradient which created a vector field from a function, the curl which generated a new vector field as its output and the divergence which outputted a scalar function given a vector field input. Those vector derivatives will appear again as we continue.

It turns out there are two ways to integrate a vector field

(1.) $\int_C \vec{F} \cdot d\vec{r}$ = line integral of $\vec{F}$ along curve C.

(2.) $\int_S \vec{F} \cdot d\vec{S}$ = surface integral of $\vec{F}$ over surface S.

The line integral is used to calculate work done by $\vec{F}$ along C and the surface integral is used calculate the net "flux" of $\vec{F}$ that "cuts" through the surface S. Of course there are many other applications, a few of which we will study. Finally, we conclude our study of vector calculus with the powerful Green's, Stokes and Divergence Th^{m}s.

Curves verses Paths

I may have inadvertently confused these two concepts before, but now we must be careful to distinguish.

Def^{n}/ A <u>curve</u> is a collection of points in $\mathbb{R}^n$, which can be parametrized by a path. If the curve has a direction we say it is <u>oriented</u>

Def^{n}/ We say $\vec{r}: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ is a <u>path</u>. It is a parametrization of the curve $r(I) \subset \mathbb{R}^n$. If C is oriented then a path $\vec{r}(t)$ is said to be <u>consistently oriented</u> if it shares the same direction.

- A path or curve is closed if it forms a loop, so $\vec{r}(a) = \vec{r}(b)$.

E137 The curve $C = \{(x,y) \mid x^2+y^2=1\}$ can be oriented in the counter-clockwise direction ↻. Its a circle. Notice

$\vec{r}_1(t) = (\cos t, \sin t)$: consistently oriented

$\vec{r}_2(u) = (\cos(-u), \sin(-u))$: oppositely oriented.

We will say $\vec{r}_2$ is consistently oriented with $-C$. That is "$-C$" means C with the opposite direction attached (↺).

- We need a few definitions to lead up to the def^n of $\int \vec{F} \cdot d\vec{r}$.

Def^n $\displaystyle\int_C f(x,y,z)\,ds \equiv \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

where we assume C is a curve with non intersecting parametrization $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$. Generally if f is a function of $(x_1, x_2, \ldots, x_n)$ then if $\vec{r}: [a,b] \to C \subset \mathbb{R}^n$,

$$\int_C f(x_1, x_2, \ldots, x_n)\,ds \equiv \int_a^b f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right| dt$$

so in $n=2$ we just drop the z-terms.

this integral is quite similar to the arclength integral which was $s = \int_a^b |r'(t)|\,dt$. The integral of f along C is calculating the area of the curtain under C, see FIGURE 2 on pg. 1071 of Stewart.

E138 Let $C = \{(x,y) \mid x^2+y^2=16,\ x \ge 0\}$. Calculate $\int_C xy^4\,ds$. We need a parametrization of C, an easy geometrically motivated choice is $x = 4\cos\theta$, $y = 4\sin\theta$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos\theta)(4\sin\theta)^4\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= (4)^6\int_{-\pi/2}^{\pi/2} \sin^4\theta\cos\theta\,d\theta$$

$$= \frac{4^6}{5}\sin^5(\theta)\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{4^6}{5}\left(1^5-(-1)^5\right) = \boxed{\frac{8192}{5}}$$

Integrals with respect to dx, dy and dz

Def'n

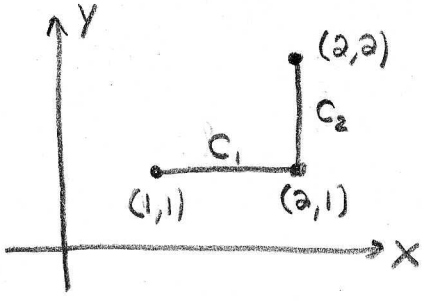
$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,\frac{dx}{dt}\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,\frac{dy}{dt}\,dt$$

$$\int_C f\,dx + \int_C g\,dy = \int_C f\,dx + g\,dy$$

- Assuming C is parametrized by the path $\vec{r}(t) = (x(t), y(t))$ which goes from $\vec{r}(a)$ to $\vec{r}(b)$.

E139 Let C be an L-shaped curve graphed below. Calculate the x-integral of $f(x,y) = xy^2$ along C. We have to break this up into pieces, I suppose our definition should have mentioned that we take the sum of the parts if we cannot globally parametrize the curve C as in here.



The curves C_1 & C_2 are easily parametrized,

$$C_1: \quad \vec{r}_1(t) = \langle 1,1\rangle + t\langle 1,0\rangle = \langle 1+t, 1\rangle, \quad 0\le t\le 1$$

$$C_2: \quad \vec{r}_2(t) = \langle 2,1\rangle + t\langle 0,1\rangle = \langle 2, 1+t\rangle, \quad 0\le t\le 1$$

Thus

$$\int_C xy^2\,dx = \int_{C_1} xy^2\,dx + \int_{C_2} xy^2\,dx$$

$$= \int_0^1 (1+t)\,\frac{d}{dt}(1+t)\,dt + \underbrace{\int_0^1 2(1+t)^2\,\frac{d}{dt}(2)\,dt}_{0}$$

this is because $dx = 0$ along C_2.

$$= \int_0^1 (1+t)\,dt$$

$$= \left(t + \tfrac{1}{2}t^2\right)\Big|_0^1$$

$$= 1 + \tfrac{1}{2} - 0$$

$$= \boxed{3/2}$$

Def^n/ The integrals along x, y or z of $f(x,y,z)$ is,

$$\int_C P\,dx + Q\,dy + R\,dz \equiv \int_a^b P(r(t))\frac{dx}{dt}dt + \int_a^b Q(r(t))\frac{dy}{dt}dt + \int_a^b R(r(t))\frac{dz}{dt}dt$$

where C is parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$.

E140 Let C be the helix $\vec{r}(t) = \langle a\cos(t), a\sin t, bt\rangle$, $0 \le t \le 2\pi$.

Lets calculate, dx, dy, dz to begin

$$x = a\cos t \qquad dx = -a\sin t\,dt$$
$$y = a\sin t \qquad dy = a\cos t\,dt$$
$$z = bt \qquad dz = b\,dt$$

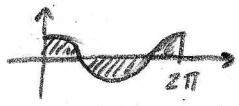
to calculate integrals over dx, dy, dz you simply convert everything to t as indicated in the def^n.

$$\int_C x\,dy - y\,dx + x\,dz = \int_0^{2\pi} (a\cos t)(a\cos t\,dt) - (a\sin t)(-a\sin t\,dt) + (a\cos t)(b\,dt)$$

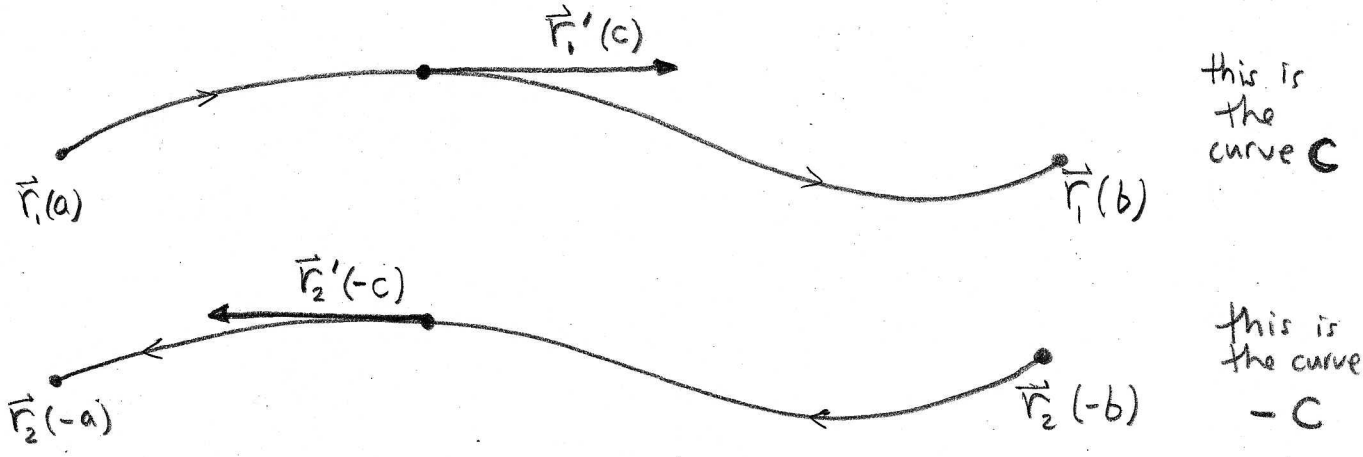
$$= \int_0^{2\pi} (a^2 + ab\cos t)\,dt$$

$$= \boxed{2\pi a^2}$$

- note $\cos t$ will average to zero over a whole period



Remark: if $\vec{r}_1(t)$ parametrizes the oriented curve C consistently we can see that $\vec{r}_2(u) = \vec{r}_1(-u)$ gives C the opposite orientation. In other words $-C$ has consistent oriented parametrization $\vec{r}_2(u)$. If $a \le t \le b$ for $\vec{r}_1(t)$ then $-b \le u \le -a$ for $\vec{r}_2(u)$.



Integrals over C verses −C

We note that the x, y, z integrals along C differ from those along $-C$ however the integral with respect to arclength is orientation independent.

$$\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx \qquad \int_{-C} f(x,y)\,dy = -\int_{C} f(x,y)\,dy \qquad \int_{C} f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

Proof: follows from the def^n and our remark on (388) $\vec{r}_2(t) = \vec{r}_1(-t)$.
$[C]$ $[-C]$

$$\int_{-C} f(x,y)\,dx = \int_{-b}^{-a} f(r_2(u))\frac{dx}{du}\,du$$

$$= \int_b^a f(r_2(-t))\frac{dx}{dt}\,dt$$

$u = -t$ and $du = -dt$

$\frac{dx}{du} = \frac{dt}{du}\frac{dx}{dt} = -\frac{dx}{dt}$

$u = -a = -t \Rightarrow t = a$

$u = -b = -t \Rightarrow t = b$

$$= -\int_a^b f(r_1(t))\frac{dx}{dt}\,dt$$

$$= -\int_C f(x,y)\,dx.$$

Notice that $ds = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du = -\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ however the other minus from switching bounds compensates this sign.

$$\int_{-C} f(x,y)\,ds = \int_{-b}^{-a} f(r_2(u))\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du$$

$$= -\int_b^a f(r_2(-t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_a^b f(r_1(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_C f(x,y)\,ds.$$

- So we know how to integrate with respect to arclength, and dx, dy, dz. Lets move on to the most important definition which happens to be related to these (in my opinion) weird integrals.

Def^n/ Let $\vec{F}$ be a continuous vector field defined on a smooth oriented curve C with parametrization $\vec{r}(t)$, $a \le t \le b$. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} \equiv \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_C \vec{F} \cdot \vec{T}\, ds$$

Let me justify that the last equality holds,

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)|\, dt \quad : \text{multiplied by one, } r(t) \text{ nonstop.}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{T}(t)\, |\vec{r}'(t)|\, dt \quad : \vec{T}(t) \equiv \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$= \int_C \vec{F} \cdot \vec{T}\, ds.$$

Now you might ask why I insisted that C is oriented. Afterall we just argued that $\int_C f(x,y)\, ds = \int_{-C} f(x,y)\, ds$. The subtle point here is that $\vec{F} \cdot \vec{T}$ depends on the orientation of the curve: $\vec{F} \cdot \vec{T}\big|_C = -\vec{F} \cdot \vec{T}\big|_{-C}$, that is the tangent vector to C points in the *opposite direction* to the tangent vector to $-C$. $\therefore$ $\boxed{\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}}$ — (*)

Remark: the line integral matches up with $\int P\,dx + Q\,dy$ in 2-d and in 3-d $\int P\,dx + Q\,dy + R\,dz$, take $\vec{F} = \langle P, Q, R \rangle$ then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q, R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\, dt$$

$$= \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

Thus $\int_C \vec{F} \cdot d\vec{r}$ and $\int_C P\,dx + Q\,dy + R\,dz$ are just alternate notations for the same concept. I prefer $\int_C \vec{F} \cdot d\vec{r}$ because it is <u>coordinate independent</u>

Examples of Line Integrals

E141 Let $\vec{F} = \langle xy, y, -yz \rangle$ and suppose the oriented curve C has parametrization $\vec{r}(t) = \langle t, t^2, t \rangle$, $0 \le t \le 1$. Calculate the line integral of $\vec{F}$ along C.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t, t^2, t) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle t^3, t^2, -t^3 \rangle \cdot \langle 1, 2t, 1 \rangle\, dt$$

$$= \int_0^1 (t^3 + 2t^3 - t^3)\, dt$$

$$= \frac{2}{4} t^4 \Big|_0^1 = \boxed{\frac{1}{2}}$$

E142 Suppose $\vec{F} = \langle xy, y, -yz \rangle$ is a *force field* and some object travels the path $\vec{r}(t) = \langle t, t^2, t \rangle$ during the time $0 \le t \le 1$. Find the *work* done by $\vec{F}$ on the object. Think infinitesimally,

$$dW = \vec{F} \cdot d\vec{r}$$

this gives the little bit of work dW done by $\vec{F}$ as the object traverses $d\vec{r}$. the dot-product is needed to pick the components of $\vec{F}$ along the direction of motion.

$$\therefore \boxed{W \equiv \int_C \vec{F} \cdot d\vec{r}} \quad -(*)$$

Therefore we have calculated the work already in **E141** it is $\boxed{W = 1/2}$.

Remark: this is a generalization of $\int F(x)\,dx$ which gave us the work done by F along x, and $W = \vec{F} \cdot \vec{d}$ which worked for *constant forces*. The definition (*) works for arbitrary forces over any old 3-d curve through space. I consider it to be the real definition of work.

More Examples:

E143 In E142 we considered $\vec{F} = \langle xy, y, -yz \rangle$ and our path went from $(0,0,0)$ to $(1,1,1)$. Suppose we let the object travel a different path between these points, say $\vec{r}(t) = \langle t, t^3, t^5 \rangle$, $0 \le t \le 1$. (C_2) What is the <u>work</u> done by $\vec{F}$ in this case?

$$W = \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \langle t^4, t^3, -t^8 \rangle \cdot \langle 1, 3t^2, 5t^4 \rangle \, dt$$

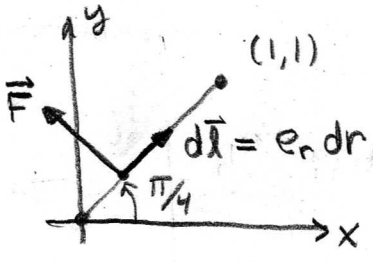
$$= \int_0^1 \left(t^4 + 3t^5 - 5t^{12} \right) dt$$

$$= \left[\frac{1}{5}t^5 + \frac{3}{6}t^6 - \frac{5}{13}t^{13} \right]_0^1$$

$$= \frac{1}{5} + \frac{1}{2} - \frac{5}{13} = \boxed{41/130}$$

<u>Remark</u>: we find that the work depends on the path taken, this is a path-dependent force. We'll see that a force must be <u>conservative</u> if the work done is to be path-independent. Notice that $\nabla \times \vec{F} \neq 0$ for this force $\therefore \nexists f$ such that $\vec{F} = \nabla f$. More on this later.

E144 Let $\vec{F} = \Theta \, e_\Theta$ where $r = \sqrt{x^2+y^2}$ and e_Θ is unit-vector in the direction of increasing Θ. Consider the radial path from $(0,0)$ to $(1,1)$, call it C.



$$\int_C \vec{F} \cdot d\vec{l} = \int_0^{\sqrt{2}} \left(\frac{\pi}{4} e_\Theta \right) \cdot (e_r \, dr)$$

$$= 0 \quad \text{since } e_\Theta \cdot e_r = 0$$

- this example illustrates the more geometrical method of evaluating path-integrals, you determine the infinitesimal line element $d\vec{l}$ then integrate as indicated.

E145 Consider $\vec{F} = \langle 0, 0, -mg \rangle = -mg\,\hat{k}$. Find the work done as the mass m travels up the slope one helix a height h. The appropriate parametrization is

$$\vec{r}(t) = \langle a\cos t, a\sin t, t \rangle \qquad 0 \le t \le h. \quad (C)$$

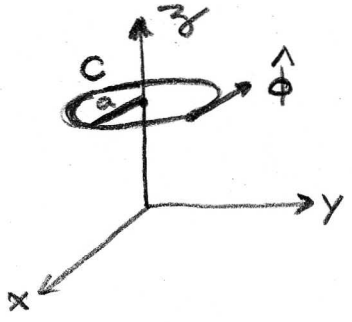
$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t, 1 \rangle = \frac{d\vec{r}}{dt}$$

$$d\vec{r} = (-a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k})\,dt$$

$$\vec{F}\cdot d\vec{r} = -mg\hat{k}\cdot(-a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k})\,dt = -mg\,dt$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^h -mg\,dt = \boxed{-mgh = W}$$

E146 Use physics cylindricals. (see 383). Suppose $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ where I, μ_0 = constant whereas $s = \sqrt{x^2+y^2}$. Let C be a circle about the z-axis with radius a at $z = z_0$, oriented by $\hat{\phi}$.



$$d\vec{\ell} = \hat{\phi}\,a\,d\phi$$

$$\vec{B}\cdot d\vec{\ell} = \frac{\mu_0 I}{2\pi s}\hat{\phi}\cdot\hat{\phi}\,a\,d\phi = \frac{a\mu_0 I}{2\pi s}d\phi$$

$$\therefore \int_C \vec{B}\cdot d\vec{\ell} = \int_0^{2\pi}\frac{a\mu_0 I}{2\pi a}d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I.$$

E147 I'll behave and write the parametrization for C in E146. Its clear that the natural parameter is ϕ, $0 \le \phi \le 2\pi$,

$$\vec{r}(\phi) = \langle a\cos\phi, a\sin\phi, z_0 \rangle$$

$$\frac{d\vec{r}}{d\phi} = \langle -a\sin\phi, a\cos\phi, 0 \rangle = a\hat{\phi}$$

$$\int_C \vec{B}\cdot d\vec{r} = \int_0^{2\pi}\vec{B}(r(\phi))\cdot\vec{r}\,'(\phi)\,d\phi$$

$$= \int_0^{2\pi}\frac{\mu_0 I}{2\pi a}\hat{\phi}\cdot a\hat{\phi}\,d\phi$$

$$= \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$

- the notation used in E146 demphasizes the parametrization. Conceptually they are the same though. The notation $d\vec{\ell}$ is $\approx$ to $d\vec{r}$.

Remark: In E147 we found the relation

$$\int_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

One physical interpretation is that this is the magnetic field $\vec{B}$ which is generated by the current $\vec{I} = I\hat{k}$ along the z-axis. This is Ampere's Law which holds for steady currents. Generally in the magnetostatic case $I = \int_S \vec{J} \cdot d\vec{A}$ where S is the surface with boundary C. When the current is not steady the situation is complicated by radiative effects... You can take a junior-level course to study E&M in depth.

Def^n/ When C is a closed path the quantity $\int_C \vec{F} \cdot d\vec{r}$ is called the circulation of $\vec{F}$ along C.

We found the circulation of $\vec{B}$ along C is proportional to the electric current. If $\vec{V}$ were the velocity field of a fluid then $\int_C \vec{V} \cdot d\vec{l}$ gives the circulation of fluid along the closed curve C, basically it measures how the fluid flows in step with the curve. If the fluid flow was everywhere perpendicular to C then the circulation would be zero. Many early advances in Electricity and Magnetism (E&M) stemmed from careful study of fluid flow and its analogy to E&M. Faraday & Maxwell had much know ledge about fluid flow while they thought out the structure of E&M. Its unfortunate the public extended the analogy to worry about electricity literally "dripping" out of wires and such, I digress.

Overall Point: Line integrals can represent many physical concepts. Geometrically, $\int_C \vec{F} \cdot d\vec{r}$ measures how the vector field $\vec{F}$ lines up with C. The more in line the bigger, if $\vec{F} \perp C$ everywhere then we get zero.