These problems are worth 1pt a piece at least. Feel free to use Mathematica or some other CAS to illustrate as needed. If I have derived a normal vector field in the notes or lecture then you may use it, but please mention the source of the fact. These problems are not the most important for Test 4, the problems on Problem Set 10 are far more on target. However, extra practice can result in a level-up.

Problem 241 Show that for a simple solid E with consistently oriented boundary ∂E if f, g are twice differentiable on some open set containing E then

$$(a.) \quad \iiint_E \left(f\nabla^2 g + \nabla f \bullet \nabla g\right) dV = \iint_{\partial E} f\nabla g \bullet d\vec{S}.$$

$$(b.) \quad \iiint_E \left(f\nabla^2 g - g\nabla^2 f\right) dV = \iint_{\partial E} \left[f\nabla g - g\nabla f\right] \bullet d\vec{S}.$$

Problem 242 Suppose $\nabla^2 f = 0$ on a simply connected solid E. If $f|_{\partial E} = 0$ then what can you say about f throughout E?
(here $|_{\partial E}$ denotes restriction of f to the subset ∂E. In particular this means you are given that $f(x, y, z) = 0$ for all $(x, y, z) \in \partial E$.)

Problem 243 Suppose $b : \mathbb{R}^3 \to \mathbb{R}$ is a particular function and $\nabla^2 f = b$ on a simply connected solid region E. If g is another such solution ($\nabla^2 g = b$) on E then show that $f = g$ on E.
The equation $\nabla^2 f = b$ is called Poisson's Equation. When $b = 0$ then it's called Laplace's Equation. You are showing the solution to Poisson's Equation is unique on a simply connected solid region. **Hint:** *use the last problem's result on $f - g$.*

Problem 244 Derive the geodesics for a sphere of radius R

Problem 245 Suppose a spring with constant k connected to a mass m is free to pivot in all directions on an inclined plane with equation $ax+by+cz = 0$. For convenience assume the spring is connected to a slide ring at the origin and the mass slides freely without friction.

Problem 246 We gave definitions for curl and divergence which were based in cartesian coordinates. Some authors actually use the identities below to define curl and divergence. Naturally, if you use these as definitions then the question of what div and curl mean are easily answered. However, on the other hand, in that approach you have no simple formula to calculate curl or div until you have mastered both surface and line integrals. I wanted to talk about curl and div before that point so for that reason I did not take these as definitions.

(a.) Assume E is a volume with piecewise smooth, outward oriented, boundary ∂E where E contains the point P. Then if we shrink the volume down to P we obtain the divergence of a differentiable $\vec{F}$ as follows:

$$div(\vec{F})(P) = \lim_{V \to 0^+} \frac{1}{V} \iint_{\partial E} \vec{F} \bullet d\vec{S}.$$

Show the formula above is true by an argument involving the divergence theorem.

(b.) Assume S is a surface with piecewise smooth, consistently oriented, boundary ∂S where E contains the point P. Then if we shrink the surface to P we obtain the curl of the the vector field in the direction of the normal $\widehat{n}$ to S at P as follows:

$$\left[curl(\vec{F})(P) \right] \bullet \widehat{n} = \lim_{A \to 0^+} \frac{1}{A} \oint_{\partial S} \vec{F} \bullet d\vec{r}$$

Problem 247 Suppose we have a vector field expressed in cylindrical coordinates; $\vec{F} = F_r \widehat{\mathbf{r}} + F_\theta \widehat{\theta} + F_z \, \widehat{\mathbf{z}}$. Calculate the formulas for

(a.) $div(\vec{F})$

(b.) $curl(\vec{F})$

By applying the boxed formulas of Problem 246 to appropriate volumes and loops. For the divergence you want to think about a little part of a cylinder which corresponds to a small change in r, θ and z. For the curl you want to think about three loops. To pick out the z-component you should fix z and allow r and θ to sweep out a little sector. I'll draw the pictures for you before class.

Problem 248 Suppose we have a vector field expressed in spherical coordinates; $\vec{F} = F_\rho \widehat{\rho} + F_\phi \widehat{\phi} + F_\theta \widehat{\theta}$. Calculate the formulas for

(a.) $div(\vec{F})$

(b.) $curl(\vec{F})$

By applying the boxed formulas of Problem 246 to appropriate volumes and loops.

Problem 249 Suppose $\vec{J} = \sigma\vec{E}$ (this is Ohm's Law for current density, the constant σ is the **conductivity**. Show that Maxwell's equations yield the equation below:

$$\nabla^2\vec{E} = \mu_o\sigma\frac{\partial\vec{E}}{\partial t} + \mu_o\epsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

also, show the same equation holds for $\vec{B}$. This is called the **telegrapher's equation**.

Problem 250 Problem 17 taken from pg. 503 of Colley's First ed. of *Vector Calculus*.

Problems 251, 252, 253, 254,255 Colley's problems 5,6,7,8,13 taken from pg. 480-482 of Colley's First ed. of *Vector Calculus.*

I am handing out a copy of 480-482 for those interested. Perhaps for the more applied students these problems from Colley will be of more interest than the deriving formulas for curl and div.

ninjas were in all these problems, however, these were too stealthy to be manifest to our untrained eye. Perhaps you will find the ninjas in the details...