These problems are worth 1pt a piece at least. Feel free to use Mathematica or some other CAS to illustrate as needed.

Problem 76 Suppose $f(x, y) = x\cosh(x + y^2)$. Calculate f_x and f_y

Problem 77 Calculate ∇f for each of the functions below:

1. $f(x, y) = 2x + 3y$

2. $f(x, y) = exp(-x^2 + 2x - y^2)$

3. $f(x, y) = \sin(x + y)$

Problem 78 What is the rate of change in the functions given in Problem 77 at the point $(1, 3)$ in the direction of the vector $\langle 1, -1\rangle$.

Problem 79 Again, concerning the functions given in Problem 77, in what directions are the functions locally constant at the point $(1, 3)$? (give answers in terms of unit-direction vectors)?

Problem 80 Find parametrizations for the normal lines through $(1, 3)$ for the functions given in Problem 77. These lines will be perpendicular to the level curve of f through $(1, 3)$.

Problem 81 find all critical points of (use integer notation for (b.) since there are many answers!)

1. $f(x, y) = exp(-x^2 + 2x - y^2)$

2. $f(x, y) = \sin(x + y)$

Problem 82 Suppose the temperature T is a function of the coordinates x, y in a large plane of battle. Furthermore, suppose the enemy ninja is carefully building a large attack by molding chakra over some time. During the preparation of the attack the enemy is vulnerable to your attack. Knowing this he has obscured the field of vision with multiple smoke bombs. However, the mass of energy building actually heats the ground. Fortunately one of your ninja skills is temperature sensitivity. You extrapolate from the temperature of the ground near your location that the temperature function has the form $T(x, y) = 50 + x - y$. In what direction should you attack?

Problem 83 [use of technology to solve algebraic and/or transcendental equation that the problem suggests] The temperature in an air conditioned room is set at 65. A ninja with expert ocular jitzu disguises himself in plain sight by bending light near him with his art. However, his art does not extend to the infared spectrum and his body heat leaves a signature variation in the otherwise constant room temperature. In particular,

$$T(x, y, z) = 33exp\big[-(x-3)^2 - (y-4)^2 - (z-1)^2\big] + 65.$$

Shino searches for the cloaked ninja by sending insect scouts which are capable of sensing a change in temperature as minute as 0.1 degree per meter. How close do the scout insects have to get before they sense the hidden ninja? (also, where is the hidden ninja and what is his body temperature on the basis of the given T which is in meters and degrees Farenheight)

Problem 84 Suppose A, B, C are constants. Calculate all nonzero partial derivatives for $z = Ax^2 + Bxy + Cy^2$.

Problem 85 Assume g, h are differentiable functions on $\mathbb{R}$. Calculate f_x and f_{xy} for

$$f(x, y) = xg(x^2 + y^2) + h(x)$$

Problem 86 The ideal gas law states that $P = kT/V$ for a volume V of gas at temperature T and pressure P. Show that

$$V\frac{\partial P}{\partial V} = -P \qquad \text{and} \qquad V\frac{\partial P}{\partial V} + T\frac{\partial P}{\partial T} = 0$$

Problem 87 The operation of $\nabla = \sum_{j=1}^{n} \widehat{\mathbf{x}}_j \frac{\partial}{\partial x_j}$ takes in a function f with domain in $\mathbb{R}^n$ and creates a vector field ∇f which assigns an n-vector at each point in $\mathbb{R}^n$. This operation has several nice properties to prove here: for differentiable real-valued functions f, g and constant c,

(a.) $\nabla(f + g) = \nabla f + \nabla g$

(b.) $\nabla(cf) = c\nabla f$

(c.) $\nabla(fg) = g\nabla f + f\nabla g$

Problem 88 Set-aside the polar coordinate notation. Define in $\mathbb{R}^n$ the spherical radius by

$$r = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} = ||\vec{r}|| = \sqrt{\vec{r} \bullet \vec{r}}$$

Show that:

(a.) $\nabla r = \frac{1}{r}\vec{r}$.

(b.) $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}$.

Problem 89 A "power" rule? Show $\nabla r^n = n r^{n-1}\widehat{r}$.

Problem 90 Find the gradient of:

1. $f(x, y, z, w) = x + y^2 + z^3 + w^4$

2. $f(x, y, z) = xyz \ln(x + y + z)$

Problem 91 Suppose Paccun speeds towards the base of a valley with paraboloid shape given by the equation $z = x^2 + 3y^2$. What is the direction of steepest **descent** at the point $(1, 1, 4)$?

Problem 92 Let $f(x, y) = x^3 - xy$. Let $A = (0, 1)$ and $B = (1, 3)$. Find a point C on the line-segment $\overline{AB}$ such that $f(B) - f(A) = \nabla f(C) \bullet (B - A)$. (*this illustrates a mean-value theorem which is known for real-valued functions of several variables*)

Problem 93 Suppose $\vec{F}(x, y, z) = \langle 2xy^2, 2x^2y, 3\rangle$. What scalar function f yields $\vec{F}$ as a gradient vector field? Find f such that $\nabla f = \vec{F}$.
(here we have to work backwards, write down what you want and guess, by the way, the function $-f$ is the pontial energy function for the force field $\vec{F}$.)

Problem 94 A basic wave equation is

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}.$$

The wave can be viewed as a graph in the xy-plane which animates with time t. Calculate appropriate partial derivatives to test if the functions give a solution to the given wave equation. ($c_1, c_2, v \neq 0$ are constants)

(a.) $y(x, t) = x - vt$

(b.) $y(x,t) = c_1 \sin(x - vt) + c_2 \cos(x - vt)$

(c.) $y(x,t) = \sin(vt)\sin(x)$

Problem 95 Show that $u = e^x \cos(y)$ and $v = e^x \sin(y)$ solves Laplaces equation $\Phi_{xx} + \Phi_{yy} = 0$.

Problem 96 Find the best linear approximation of each object at the given point. Also, write either and equation or a parametrization of the tangent space in each case ("space" could mean line, surface, space curve or other things...)

(a.) $f(x) = x^2$ at $a = 2$

(b.) $f(x, y) = x^2 - 2xy$ at $(3, 4)$

(c.) $\vec{r}(t) = \langle t, 3, t^2 2^t \rangle$ at $t = 0$

Problem 97 Given that $x = u^2 + v^2$ and $y = 3uv$ and $z = 3\sin(uv)$ and $w = ze^{xy}$ calculate w_u and w_v and finally w_z.

Problem 98 Calculate the Jacobian matrix of $\vec{r}(u, v) = \langle u^2+v^2,\ 3uv,\ 3\sin(uv)\rangle$ and that of $f(x, y, z) = ze^{xy}$. Multiply these matrices and identify how this relates to the previous problem.

Problem 99 Suppose $z = xy$ and $x = \sinh[g(t)]$ and $y = h(t^2)$ for some differentiable functions g, h. Calculate dz/dt by the chain rule(s).

Problem 100 Suppose a car speeds over a hill with equation $x^2 + 4y^2 + z^2 = 1$. If at the point with $x = 1m$ and $y = 0.5m$ the car has an x-velocity of $10m/s$ and a y-velocity of $20m/s$ then what z-velocity does the car have? (assume the car stays on hill)