These problems are worth 1pt a piece at least. Feel free to use Mathematica or some other CAS to illustrate as needed.

Problem 101 Suppose S is the surface defined by $F(x,y,z) = xyz = 1$. Find the equation of the tangent plane and the parametrization of the normal line through $(1,1,1)$.

Problem 102 Find a parametrization $\vec{X}$ of S from the previous problem which provides a patch in the locality of $(1,1,1)$. Use α, β for your parameters and find the normal-vector field $\vec{N}(\alpha,\beta)$ by computing $\vec{N}(\alpha,\beta) = \frac{\partial \vec{X}}{\partial \alpha} \times \frac{\partial \vec{X}}{\partial \beta}$. Do you obtain the same normal vector at $(1,1,1)$ with this patch?

Problem 103 Label the solution set of $x^2 = y - z^2$ as M.

(a.) present M as a level-surface for some function F. Explicitly state the formula for F. Find the normal vector field on M.

(b.) parametrize M and once more find the normal vector field. This time find $\vec{N}$ explicitly in terms of your chosen parameters.

Problem 104 Suppose a level surface $G(x,y,z) = 2$ has $\nabla G(x,y,z) = \langle x, y, z\rangle$ and another level surface $F(x,y,z) = 42$ with $\nabla F(x,y,z) = \langle 1, x, 3\rangle$. Suppose these surfaces intersect along some curve and at the point (a,b,c) the curve of intersection has tangent line with direction vector colinear to $\langle 0, 0, 10\rangle$. Find (a,b,c).

Problem 105 Consider the line-segment $\mathcal{L} = \overline{PQ}$ where $P = (1, 2, 3)$ to $Q = (5, 0, -1)$.

(a.) describe $\mathcal{L}$ parametrically as a path from $t = 0$ at P to $t = 1$ at Q.

(b.) describe $\mathcal{L}$ parametrically as a path from $s = 0$ at Q to $s = 6$ at P.

(c.) describe $\mathcal{L}$ as a graph. In particular, find $h(x)$ and $g(x)$ such that $f(x) = \langle g(x), h(x) \rangle$ and $graph(f) = \{(x, f(x)) \mid x \in dom(f)\ \}$.

(d.) describe $\mathcal{L}$ as a level-curve. In particular, find F such that $\mathbb{R}^3 \xrightarrow{F} \mathbb{R}^2$ and $\mathcal{L} = F^{-1}\{(0, 0)\}$.

Remark: personally, I view the last two parts of the previous problem as less natural than the parametric presentation. It is in fact possible to present lines, surfaces, volumes etc... as either graphs, level-sets or as parametrized objects. Which to use depends on the context.

Problem 106 Ohms' Law says that $V = IR$ where V is the voltage of a battery which delivers a current I to a resistor R. As the current flows the battery will wear down and the voltage will drop. On the other hand, as the resitor heats-up the resistance will increase. Given that $R = 600$ ohms and $I = 0.04$ amp, if the resistance is increasing at a rate of 0.5 ohm/sec and the voltage is dropping at 0.01 volt/sec then what is the rate of change in the current I at this time.

Problem 107 Suppose that the temperature T in the xy-plane changes according to

$$\frac{\partial T}{\partial x} = 8x - 4y \qquad \& \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

Find the maximum and minimum temperatures of T on the unit circle $x^2 + y^2 = 1$. To achieve this goal you should parametrize the the circle by $x = \cos t$ and $y = \sin t$ and calculate dT/dt and d^2T/dt^2 by the chain-rule. (you have no other option since the formula for T is not given!)

Problem 108 Suppose $f(u, v, w)$ is the formula for a differentiable f and $u = x - y$, $v = y - z$ and $w = z - x$. Show that $f_x + f_y + f_z = 0$.

Problem 109 Suppose $w = xy^2 + z^3$ and $x = f(u, v)$, $y = g(u, v)$ and $z = h(u, v)$ where f, g, h are differentiable functions. If $f(1, 2) = 2$ and $g(1, 2) = 3$ and $h(1, 2) = 4$ and $g_v(1, 2) = 0$, $h_v(1, 2) = 7$ and $f_v(1, 2) = 42$ calculate $\frac{\partial w}{\partial v}(1, 2)$.

Problem 110 Suppose $f(x,y) = x^2 - 3xy + 5$. A theorem states that for twice continuously differentiable f the error $E(x,y) = f(x,y) - L(x,y)$ in the linearization for each (x,y) in some rectangle R centered at (x_o, y_o) is bounded by

$$|E(x,y) \leq \frac{1}{2}M\bigg(|x - x_o + |y - y_o|\bigg)^2$$

where M bounds $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ on R. In other words, if you can find such an M to bound the second order partial derivatives then the error is given by the inequality above.

(a.) find the linearization of f at $(2,1)$.

(b.) bound the error $E(x,y)$ for the square $[1.9, 2.1] \times [0.9, 1.1]$

Remark: not that I plan to derive it this semester, but this is a consequence of the error estimate for the single-variable Taylor series as it applies to the construction of the multivariate Taylor expansion. The multivariate Taylor expansion derives from the chain-rule and Taylor's theorem from calculus II.

Problem 111 The area of a triangle is given by $A = \frac{1}{2}ab\sin\gamma$ where a, b are the lengths of two sides which have angle γ between them. Suppose that $\gamma = \pi/3 \pm 0.01$ and $a = 150 \pm 1$ ft and $b = 200 \pm 1$ ft. Find the corresponding uncertainty in the area. (leave answer as $A \pm \delta$ where the δ is calculated from the differential of A)

Problem 112 Suppose you know $x = 3 \pm 0.01$ and $y = 4 \pm 0.01$ what are the corresponding polar coordinate ranges.

Problem 113 The kinetic energy in 2D-problem with cartesian coordinates is given by $K = \frac{1}{2}mv^2$ or explicitly in terms of the x, y velocities $\dot{x}, \dot{y}$ we have $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. Calculate the formula for K in terms of polar coordinates r, θ and their velocities $\dot{r}, \dot{\theta}$.

Problem 114 Suppose $w = x^2 + y - z + \sin(t)$ and $x + y = t$. Calculate the following constrained partial derivatives:

(**a**.) $\left(\frac{\partial w}{\partial y}\right)_{x,z}$ (**b**.) $\left(\frac{\partial w}{\partial y}\right)_{z,t}$ (**c**.) $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ (**d**.) $\left(\frac{\partial w}{\partial z}\right)_{y,t}$

Problem 115 Suppose $\vec{F} = \rho^2\,\widehat{\rho} + \frac{1}{\rho}\theta^3\sin(\phi)\,\widehat{\phi}$ find f such that $\nabla f = \vec{F}$.

Problem 116 Suppose $f(x,y,z) = (x^2+y^2+z^2)\tan^{-1}(y/x) + \cos^{-1}(z/\sqrt{x^2+y^2+z^2})$. Calculate ∇f.

Problem 117 Suppose a formula for $f(x,y)$ is given. Furthermore, suppose you are asked to calculate $\frac{\partial f}{\partial r}$ where $r = \sqrt{x^2+y^2}$. Techincally, this question is ambiguous. Why? Because you need to know what other variable besides r is to be used in concert with r. If we use the usual polar coordinates then $\tan(\theta) = \frac{y}{x}$ and all is well. We adopt the following (standard) interpretation:

$$f_r = \frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\Big[f(r\cos\theta, r\sin\theta)\Big] = \frac{\partial f}{\partial x}\bigg|_{(r\cos\theta, r\sin\theta)}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\bigg|_{(r\cos\theta, r\sin\theta)}\frac{\partial y}{\partial r}$$

In other words, we define the derivative of f with respect to some curvelinear coordinate by the derivative of $f \circ \vec{T}$ where $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is the coordinate transformation to which the curvelinear coordinate belongs. Denoting $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$ we define,

$$f_\theta = \frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\Big[(f\circ\vec{T})(r,\theta)\Big] = \frac{\partial f}{\partial x}\bigg|_{(r\cos\theta, r\sin\theta)}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\bigg|_{(r\cos\theta, r\sin\theta)}\frac{\partial y}{\partial \theta}$$

A short calculation reveals that:

$$f_r = f_x\cos\theta + f_y\sin\theta \qquad \& \qquad f_\theta = -f_x r\sin\theta + f_y r\cos\theta$$

Solve the equations above for f_x and f_y.

Problem 118 Recall that $\nabla \bullet \nabla\Phi = \nabla^2\Phi = 0$ is called Laplace's equation. In cartesian coordinates, in two dimensions, Laplace's equation reads $\Phi_{xx} + \Phi_{yy} = 0$. Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0.$$

(yes, most of this is in the notes, but I'd like to see the rest of the details)

Problem 119 Given the potential functions Φ below show they are solutions to Laplace's equations either via computation in cartesian coordinates or polar coordinates.

(a.) $\Phi(x, y) = \sqrt{x^2 + y^2}$

(b.) $\Phi(x, y) = \tan^{-1}(y/x)$

(c.) $\Phi(r, \theta) = r^2 \cos\theta \sin\theta$

Problem 120 Define hyperbolic coordinates h, ϕ by the following equations

$$x = h\cosh\phi \qquad \& \qquad y = h\sinh(\phi)$$

Let's study these coordinates by answering the following:

(a.) solve the equations above for h and ϕ.

(b.) Find hyperbolic coordinates for $(1,1), (-1,1), (-1,-1)$ and $(1,-1)$. Write a diagram which explains the signs for h and ϕ in each quadrant.

(c.) What do are level curves of h ?

(d.) What are level curves of ϕ?

Problem 121 Continuing the study from the previous problem,

(a.) find functions A, B, C, D of hyperbolic coordinates h, ϕ which give unit-vectors

$$\widehat{h} = A\,\widehat{\mathbf{x}} + B\,\widehat{\mathbf{y}} \qquad \& \qquad \widehat{\phi} = C\,\widehat{\mathbf{x}} + D\,\widehat{\mathbf{y}}$$

(b.) derive a formula for ∇f in terms of hyperbolic coordinate derivatives, however express it in terms of $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$. That is find $\nabla f = \langle f_x, f_y \rangle$ but express f_x and f_y in terms of the hyperbolic coordinates and derivatives.

(c.) derive ∇f is purely hyperbolic notation: that is find E, F such that

$$\nabla f = E\widehat{h} + F\widehat{\phi}.$$

partial derivative computation is fun... but, what does it mean? We explore this question in the pair of problems below

Problem 122 Let $(a, b) \in \mathbb{R}^2$ be a particular point. Explain geometrically the meaning of the equations given below:

(a.) $\frac{\partial f}{\partial r}(a, b) = -1$

(b.) $\frac{\partial f}{\partial \theta}(a, b) = 1$

(c.) $\frac{\partial f}{\partial \phi}(a, b) = 0$ (same notation as in previous pair of problems)

As an example: $\frac{\partial f}{\partial x}(a, b) = 0$ indicates that the function stays constant along the line passing through (a, b) on which y is held fixed at value b (parametrically f is constant along the path $t \to (a + ta, b)$ near $t = 0$).

Problem 123 Joshua asked if $\frac{\partial}{\partial(xy)}$ had meaning. I would say yes. In fact, it has many meanings.

(a.) Define $u = xy$ and $v = y/x$ for $(x, y) \in (0, \infty)^2$. Find inverse transformations. That is, solve for $x = x(u, v)$ and $y = y(u, v)$ in view of the definition just given and comment on the level curves of u, v (if they are a named curve then name them).

(b.) explain what $\frac{\partial f}{\partial u} = 0$ means for a function f at a given point. (use meaning suggested from part (a.))

(c.) Define $u = xy$ and $w = y$ for $(x, y) \in (0, \infty)^2$. Find inverse transformations. That is, solve for $x = x(u, w)$ and $y = y(u, w)$ in view of the definition just given and comment on the level curves of u, w (if they are a named curve then name them).

(d.) explain what $\frac{\partial f}{\partial u} = 0$ means for a function f at a given point. (use meaning suggested from part (c.)) (it is **not** a directional derivative in the traditional sense of the term.

the previous problem is important for applications. Think about this, what variable is most interesting to your model? It is important to be able to write the equations which describe the model in terms of those variables. On the other hand, it may be simple to express the physics of the model in cartesian coordinates. Hopefully these problems give you an idea about how to translate from one formalism to the other and vice-versa.

Problem 124 Suppose that the temperature T in the xy-plane changes according to

$$\frac{\partial T}{\partial x} = 8x - 4y \qquad \& \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

Find the maximum and minimum temperatures of T on the unit circle $x^2 + y^2 = 1$. This time use the method of Lagrange multipliers. Hopefully we find agreement with Problem 107.

Problem 125 Use the method of Lagrange multipliers to find the point on the plane $x + 2y - 3z = 10$ which is closest to the point $(8, 8, 8)$.