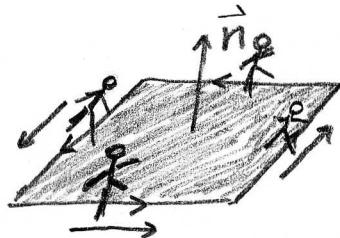
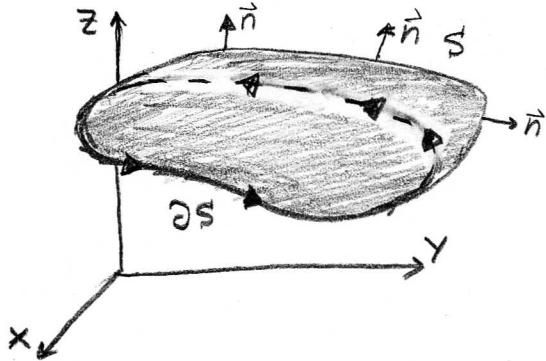


STOKES' THEOREM

(4/2)



S is an oriented surface
 ∂S is boundary with orientation induced from S .

- If we walk around ∂S in the positive direction with our head on the normal's side then S will be on our left always.
- You can also use the right hand rule. Point your thumb along \vec{n} then your fingers indicate the direction of ∂S .

Th^m(STOKES') Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve ∂S with positive induced orientation from S . Let F be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Proof: See Stewart.

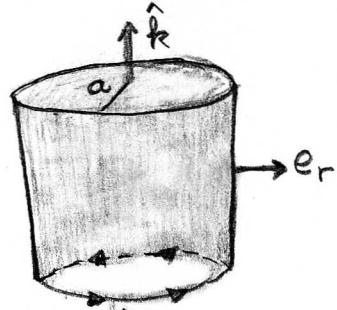
E164 Notice that $\vec{A} = -\frac{\mu_0 I}{2\pi r} \ln(r) \hat{k}$ has $A_z = \frac{-\mu_0 I}{2\pi r} \ln(r)$

$$\nabla \times \vec{A} = -\frac{\partial A_z}{\partial r} e_\theta = \frac{\mu_0 I}{2\pi} \frac{\partial [\ln(r)]}{\partial r} e_\theta = \frac{\mu_0 I}{2\pi r} e_\theta = \vec{B} \text{ from E161.}$$

where I have used the Th^m from (380) to compute the curl in cylindricals.
 It's clear that if we let S be the cylinder without the base, then

from the calculation in E161 we have $\iint_S \vec{B} \cdot d\vec{A} = 0$.

Lets check if Stoke's Th^m agrees



∂S with
induced orientation
notice $d\vec{l} = (ad\theta) e_\theta$

$$\begin{aligned} \iint_S \vec{B} \cdot d\vec{A} &= \iint_S (\nabla \times \vec{A}) \cdot d\vec{A} \\ &= \int_{\partial S} \vec{A} \cdot d\vec{l} \\ &= \int_{\partial S} -\frac{\mu_0 I}{2\pi r} \ln(r) \hat{k} \cdot (ad\theta) e_\theta = 0 \end{aligned}$$

↑ orthogonal.

Remark: The function \vec{A} that gives \vec{B} by $\vec{B} = \nabla \times \vec{A}$ is called the "vector potential". Its the analogue of the scalar potential V that gives \vec{E} by $\vec{E} = -\nabla V$. Actually in general you need both to get \vec{E} and \vec{B} ... (not req'd topic)

E165 GREENE'S THEOREM: (§17.4)

An application of Stokes' Th^m to the xy-plane was discovered before Stokes' Th^m historically. However, we may view it as an elementary application of Stokes' Th^m in the following case. Let D be a region bounded by C a simple closed curve, that is let D be a simply connected region with boundary ∂D and suppose D lies in xy-plane. Consider continuous vector field $\vec{F} = \langle P, Q, 0 \rangle$ with continuous partial derivatives of P, Q ,

$$\iint_D (\nabla \times \vec{F}) \cdot d\vec{s} = \oint_{\partial D} \vec{F} \cdot d\vec{r}$$

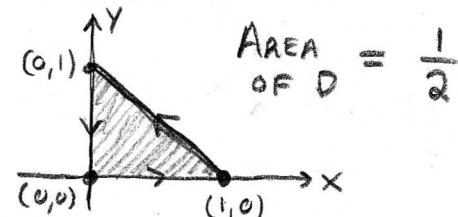
here D has $ds = dx dy \hat{k}$ and ∂D is oriented counter-clockwise, then since $(\nabla \times \vec{F}) \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ we find (recall Remark on 390)

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P dx + Q dy$$

In the context of \mathbb{R}^2 this applies to $\vec{F} = \langle P, Q \rangle$.

E166 Lets apply Green's Th^m to the triangular region D formed by the points $(0,0), (1,0), (0,1)$. Let C be the boundary of the triangle oriented CCW, calculate the line integral via Green's Th^m,

$$\begin{aligned} \oint_C -y dx + x dy &= \iint_D \left(\frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[-y] \right) dA : \vec{F} = \langle -y, x \rangle, \text{ use Green's Th}^m \\ &= \iint_D 2 dA \\ &= 2 \text{ Area}(D) \\ &= 1 \end{aligned}$$



- We see Green's Th^m allows us to trade a line integral for a surface integral. This is advantageous if one or the other is easy to calculate, like in the example above.

E167 If we choose a vector field $\vec{F} = \langle P, Q \rangle$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then this will give us another method to calculate the area of D .

$$\text{Area}_D = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P dx + Q dy.$$

So are there such vector fields? Yes, for example

$$\vec{F} = \langle 0, x \rangle \quad \text{or} \quad \vec{F} = \langle -y, 0 \rangle \quad \text{or} \quad \vec{F} = \frac{1}{2} \langle -y, x \rangle$$

This gives us the following sneaky formulas for area,

$$\boxed{\text{Area}(D) = \int_{\partial D} x dy = \int_{\partial D} -y dy = \frac{1}{2} \int_{\partial D} x dy - y dx} \quad (*)$$

Then consider the ellipse $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. Then ∂D has parametrization for $0 \leq \theta \leq 2\pi$

$$\begin{aligned} x &= a \cos \theta & \Rightarrow dx = -a \sin \theta d\theta \\ y &= b \sin \theta & \Rightarrow dy = b \cos \theta d\theta \end{aligned}$$

Thus using $(*)$,

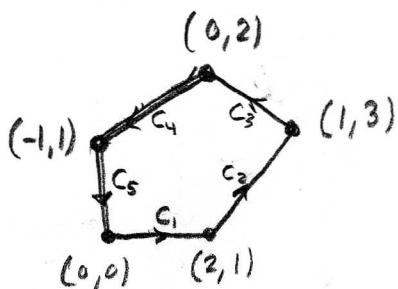
$$\begin{aligned} \text{Area}(D) &= \int_{\partial D} x dy \\ &= \int_0^{2\pi} (a \cos \theta)(b \cos \theta d\theta) \\ &= ab \int_0^{2\pi} \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= ab \left[\frac{\theta}{2} \Big|_0^{2\pi} + \frac{ab}{4} \sin(2\theta) \Big|_0^{2\pi} \right] \\ &= \boxed{\pi ab} \end{aligned}$$

E168 Let C be a line segment from (x_1, y_1) to (x_2, y_2) then we may parametrize $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$ ($0 \leq t \leq 1$)

$$\begin{aligned} \int_C x dy - y dx &= \int_0^1 [(x_1 + t(x_2 - x_1))[(y_2 - y_1)dt] - [y_1 + t(y_2 - y_1)][(x_2 - x_1)dt]] \\ &= \int_0^1 [x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(x_2 - x_1)(y_2 - y_1) - (y_2 - y_1)(x_2 - x_1)]] dt \\ &= \int_0^1 [x_1 y_2 - \cancel{x_1 y_1} - y_1 x_2 + \cancel{y_1 x_1}] dt \\ &= x_1 y_2 - y_1 x_2. \end{aligned}$$

E168 Continued We found a simple formula for $\int_C x dy - y dx$

in the case C is a line segment, this coupled to Green's Thⁿ gives a simple method to calculate the area of pentagons, octagons, etc... Consider the pentagon P ,



$$\int_{C_1} x dy - y dx = x_1 y_2 - x_2 y_1 = 0$$

$$\int_{C_2} x dy - y dx = 2(3) - (1)(1) = 5$$

$$\int_{C_3} x dy - y dx = 2 - 0 = 2$$

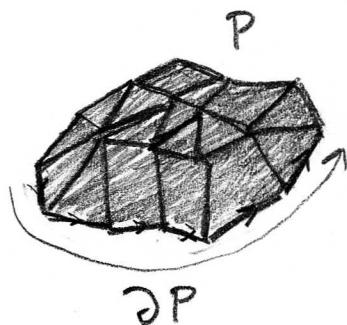
$$\int_{C_4} x dy - y dx = 0(-1) - (-1)(2) = 2$$

$$\int_{C_5} x dy - y dx = -1(0) - (0)(1) = 0$$

Then notice that P has boundary $\partial P = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$.

$$\begin{aligned} \text{Area}(P) &= \frac{1}{2} \int_{\partial P} x dy - y dx \quad \text{using (x) on 414} \\ &= \frac{1}{2} [0 + 5 + 2 + 2 + 0] \\ &= \boxed{\frac{9}{2}} \end{aligned}$$

Remark: this is interesting in the context of the xy -plane and Green's Thⁿ, but this trick will also allow you to calculate the surface area of rather ugly polyhedra.



$$\iint_P (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial P} \vec{F} \cdot d\vec{l}$$

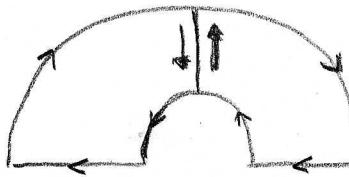
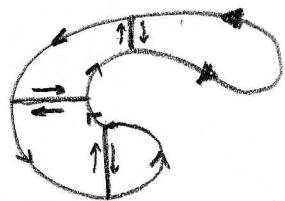
The surface integral of $\nabla \times \vec{F}$ over P can be calculated from a line integral of \vec{F} around ∂P . If ∂P lies in the xy -plane we can use an approach similar to that of E168

(one catch, need $\iint_P (\nabla \times \vec{F}) \cdot d\vec{S} = (\nabla \times \vec{F}) \cdot \iint_P d\vec{S}$ could be hard to find such an \vec{F})

REGIONS WITH HOLES, How to APPLY GREEN's Th²

416

According to Stewart TYPE I and TYPE II regions are simple regions. Green's Th² can be extended to other regions, let's draw a few pictures to illustrate

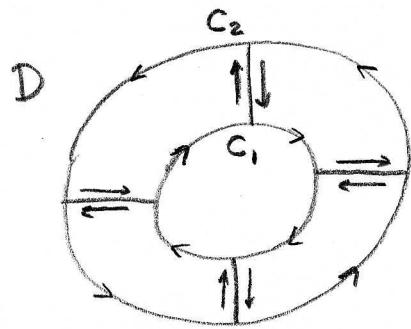


Just like simple regions

$$\iint_S (\partial_x Q - \partial_y P) dA = \int_{\partial S} P dx + Q dy$$

Just divide the shape up into simple regions and add the results together, the places where the divisions share a side do not contribute because the sides have opposing orientations and as such cancel out, thus only the outer edge of the shape matters, or in the case of a donut the inner edge also matters and it

must be given CW orientation in contrast to the CCW orientation of the outer edge. Notice each subregion has a CCW orientation.

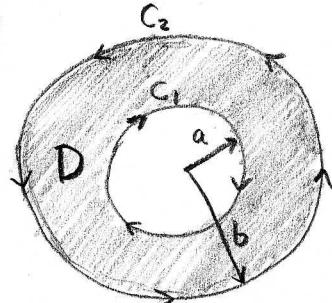


$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy$$

For regions like this donut we must take care to orient the inner loops in a CW fashion.

Remark: if my pictures here aren't convincing take a look at § 17.4 of Stewart.

E169



$$C_1: \begin{cases} x = a \cos t \\ y = -a \sin t \end{cases} \quad C_2: \begin{cases} x = b \cos t \\ y = b \sin t \end{cases} \quad 0 \leq t \leq 2\pi$$

Choose $\vec{F} = \langle 0, x \rangle$ so $\partial_x Q - \partial_y P = 1$ thus

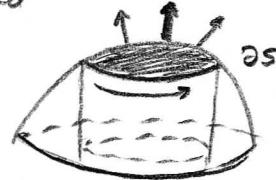
$$\begin{aligned} A &= \iint_D dA = \int_0^{2\pi} (a \cos t)(-a \sin t) dt + \int_0^{2\pi} (b \cos t)(b \sin t) dt \\ &= \int_0^{2\pi} (b^2 - a^2) \frac{1}{2}(1 + \cos 2t) dt = \boxed{\pi b^2 - \pi a^2} \end{aligned}$$

EXAMPLES OF STOKE'S TH^M

(417)

E170 Let $\vec{F} = \langle y_3, x_3, xy \rangle$ and let S be the ($z > 0$) surface $x^2 + y^2 + z^2 = 4$ bounded by $x^2 + y^2 = 1$. Compute the integral $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$. Use Stoke's Th^M and we'll not even need to take the curl of \vec{F} , instead

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$



The boundary of S is ∂S , it satisfies

$$x^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4$$

$$\Rightarrow 1 + z^2 = 4$$

$$\Rightarrow z = \pm \sqrt{3}$$

$\Rightarrow z = \sqrt{3}$ since we assume $z > 0$ in the statement of the problem.

We parametrize ∂S by $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} x &= \cos \theta & y &= \sin \theta & z &= \sqrt{3} \\ dx &= -\sin \theta d\theta & dy &= \cos \theta d\theta & dz &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \int_{\partial S} \vec{F} \cdot d\vec{r} &= \int_{\partial S} F_1 dx + F_2 dy + F_3 dz \\ &= \int_0^{2\pi} (\sqrt{3} \sin \theta)(-\sin \theta d\theta) + (\sqrt{3} \cos \theta)(\cos \theta d\theta) + (\sin \theta \cos \theta)(0) \\ &= \int_0^{2\pi} \sqrt{3} (\cos \theta \cos \theta - \sin \theta \sin \theta) d\theta \\ &= \int_0^{2\pi} \sqrt{3} \cos(2\theta) d\theta \\ &= \frac{\sqrt{3}}{2} \sin 2\theta \Big|_0^{2\pi} \\ &= 0. \quad \therefore \boxed{\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0} \end{aligned}$$

E171 Let S be the rectangle on $Z=y$ plane that projects to $0 \leq x \leq 1$, $0 \leq y \leq 3$. Suppose

$$\vec{F} = \langle x^2, 4xy^3, y^2x \rangle$$

(*) calculate $\int_S \vec{F} \cdot d\vec{r}$. I'll use Stokes Thm to convert to surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & y^2x \end{vmatrix} = \langle 2yx, -y^2, 4y^3 \rangle$$

We'll need to parametrize S' in order to compute the surface integral. May I recommend x & y as parameters,

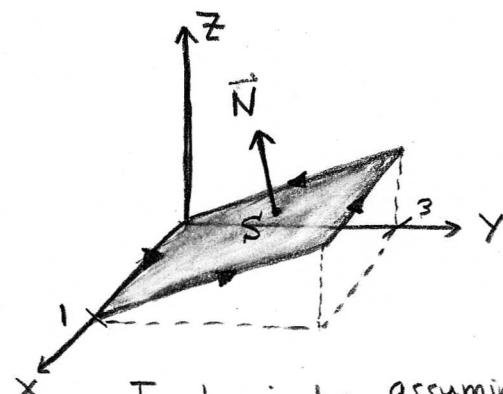
$$\vec{r}(x, y) = \langle x, y, y \rangle, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 3 \quad \text{call this } D.$$

$$\vec{r}_x \times \vec{r}_y = \langle 1, 0, 0 \rangle \times \langle 0, 1, 1 \rangle = \hat{k} - \hat{j} \quad (\text{not surprising, think about } z-y=0)$$

$$\therefore \vec{N}(x, y) = \langle 0, -1, 1 \rangle$$

Thus,

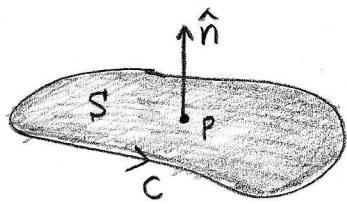
$$\begin{aligned} \int_S \vec{F} \cdot d\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}, \quad \text{note } ds = \langle 0, -1, 1 \rangle dx dy \\ &= \iint_D \langle 2yx, -y^2, 4y^3 \rangle \cdot \langle 0, -1, 1 \rangle dA \\ &= \int_0^3 \int_0^1 (y^2 + 4y^3) dx dy \\ &= \int_0^3 (y^2 + 4y^3) dy \\ &= \left(\frac{1}{3}y^3 + y^4 \right) \Big|_0^3 \\ &= \frac{1}{3}(27) + 81 \\ &= 90 \end{aligned}$$



I begin by assuming dS is oriented as pictured. Then to be consistent we must choose \vec{N} as pictured.

Remark: this example is borrowed from Howard Anton's "CALCULUS" 5th Ed. See E2 pg. 978. This is a nice book, lots of history.

Remark: Other texts actually define the curl by a limiting integral condition. This allows an alternative method of calculating the curl in curvilinear coordinates.



$$\hat{n} \cdot (\nabla \times \vec{F})(P) = \lim_{A \rightarrow 0} \frac{1}{A} \oint_C \vec{F} \cdot d\vec{r}$$

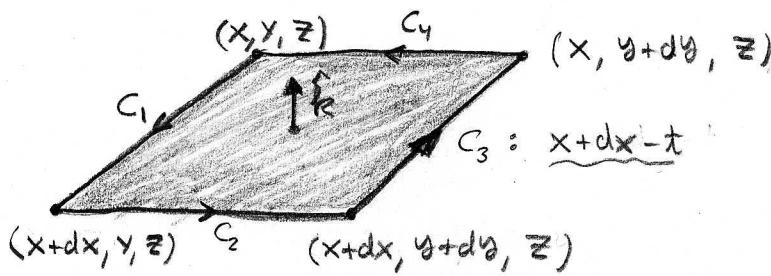
gives the component of $\nabla \times \vec{F}$ along \hat{n} .

For an infinitesimal area $d\vec{A}$ we can use Stoke's Th^m, $d\vec{A} = (dA)\hat{n}$

$$\iint_{d\vec{A}} (\nabla \times \vec{F}) \cdot d\vec{s} = \oint_{\partial(d\vec{A})} \vec{F} \cdot d\vec{r}$$

$$\iint_{d\vec{A}} (\nabla \times \vec{F}) \cdot d\vec{A} = \oint_{\partial(d\vec{A})} \vec{F} \cdot d\vec{r} \Rightarrow (\nabla \times \vec{F}) \cdot \hat{n} = \frac{1}{dA} \oint_{\partial(d\vec{A})} \vec{F} \cdot d\vec{r}$$

Let $\vec{F} = \langle P, Q, R \rangle$ and $d\vec{A} = dx dy \hat{k}$.



$$\partial(d\vec{A}) = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$C_1 : dx = dt, dy = dz = 0$$

$$C_2 : dy = dt, dx = dz = 0$$

$$C_3 : dx = -dt, dy = dz = 0$$

$$C_4 : dy = -dt, dx = dz = 0$$

$$(\nabla \times \vec{F})_{z=0} dx dy = \oint_{\partial(d\vec{A})} \vec{F} = \int_{C_2} P dx + Q dy + R dz = [Q(x+dx, y, z) - Q(x, y, z)] dy - [P(x, y+dy, z) - P(x, y, z)] dx$$

$$\Rightarrow (\nabla \times \vec{F}) \cdot \hat{k} = \frac{Q(x+dx, y, z) - Q(x, y, z)}{dx} - \frac{P(x, y+dy, z) - P(x, y, z)}{dy}$$

$$(\nabla \times \vec{F})_3 = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Remark: Notice we didn't assume that $\nabla \times \vec{F}$ had this form, we derived it here. Personally I'm not much of a fan of this approach, if you'd like to see more check out problems #23 & 24 of §7.3 (Calley 1st Ed.)