

# PARAMETRIC SURFACES & THEIR NORMAL VECTOR FIELD

402

We discussed the idea of a parametric surface & its tangent plane on 317-319. The surface integral of a vector field is phrased in terms of a parametric description of the surface, it is thus important for us to review what a parametric surface is.

**Defn** A parametrized surface  $S$  consists of a parameter space  $D \subseteq \mathbb{R}^2$  and a mapping  $\vec{\Sigma}$  which is mostly invertible  $\vec{\Sigma}: D \rightarrow S \subset \mathbb{R}^3$ ,  $(u, v) \mapsto \vec{\Sigma}(u, v)$ . Where

$$\vec{\Sigma}(u, v) = (x(u, v), y(u, v), z(u, v))$$

We say  $S$  is oriented if the normal vector field  $\vec{N}(u, v)$  is nonzero  $\forall (u, v) \in D$ , where

$$\vec{N}(u, v) = \frac{\partial \vec{\Sigma}}{\partial u} \times \frac{\partial \vec{\Sigma}}{\partial v}$$

We may also use  $\vec{r}(u, v)$  as the mapping in which case the normal vector field would be  $\vec{N}(u, v) = \vec{r}_u \times \vec{r}_v$ .

**E151**  $\vec{\Sigma}(\theta, \varphi) = R(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$  with  $R > 0$  and  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \varphi \leq \pi$  parametrizes a sphere. (see E78 317)

$$\frac{\partial \vec{\Sigma}}{\partial \theta} = R \langle -\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0 \rangle$$

$$\frac{\partial \vec{\Sigma}}{\partial \varphi} = R \langle \cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi \rangle$$

$$\vec{N}(\theta, \varphi) = R^2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \theta \sin \varphi & \cos \theta \sin \varphi & 0 \\ \cos \theta \cos \varphi & \sin \theta \cos \varphi & -\sin \varphi \end{vmatrix}$$

$$= R^2 \langle -\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin^2 \theta \sin \varphi \cos \varphi - \cos^2 \theta \sin \varphi \cos \varphi \rangle$$

$$= R^2 \sin \varphi \langle -\cos \theta \sin \varphi, -\sin \theta \sin \varphi, -\cos \varphi \rangle \quad \text{see (*) on 364 to get}$$

$$= -(R^2 \sin \varphi) e_p$$

$$(e_p = \vec{e}_p / |\vec{e}_p|)$$

$$\therefore \vec{N}(\theta, \varphi) = -(R^2 \sin \varphi) e_p$$

this is the normal vector field to the sphere, since  $\sin \varphi \geq 0$  for  $0 \leq \varphi \leq \pi$  we note that  $\vec{N}(\theta, \varphi)$  points towards origin.

**Def<sup>n</sup>** A closed surface  $S'$  is the boundary of some solid region  $E$ ;  $S = \partial E$ . If the normal vector field points outward everywhere on  $S$  then we say  $S'$  is positively oriented, if it points inward everywhere we say  $S'$  has a negative orientation.

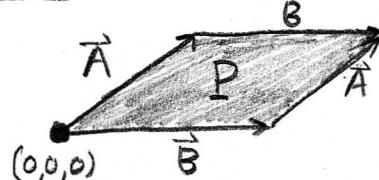
**Remark:** If  $\vec{\Sigma}(u, v)$  is negatively oriented then  $\vec{\Sigma}(v, u)$  is positively oriented. Switching the order of the parameters will switch  $\vec{N}(u, v) = \vec{\Sigma}_u \times \vec{\Sigma}_v$  to  $\vec{N}(v, u) = \vec{\Sigma}_v \times \vec{\Sigma}_u$ .

**E152** We found  $\vec{\Sigma}(\theta, \varphi) = R(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$  yields a negative orientation to the sphere since  $\vec{N}(\theta, \varphi) = -R^2 \sin \varphi \mathbf{e}_p$  points inward. To get a positive orientation for the sphere we need only switch the order of  $\theta$  and  $\varphi$ . Then

$$\vec{N}(\varphi, \theta) = \vec{\Sigma}_\varphi \times \vec{\Sigma}_\theta = -\vec{\Sigma}_\theta \times \vec{\Sigma}_\varphi = R^2 \sin \varphi \mathbf{e}_p = \vec{N}(\varphi, \theta)$$

**Remark:** For many surfaces we cannot intrinsically choose a normal vector, I mean there are two choices. The plane is a good example, both  $\langle a, b, c \rangle$  and  $\langle -a, -b, -c \rangle$  provide a normal vector field, which should we choose? The answer is simple, we cannot choose, sometimes to make the problem unambiguous we must be told both the surface and the direction of its orientation. However, if we are given  $\vec{\Sigma}(u, v)$  then  $\vec{N}(u, v)$  is implicitly given.

**E153** Find a parametrization for the parallelogram pictured below.



$$\vec{\Sigma}(s, t) = s\vec{A} + t\vec{B} \quad \text{for } 0 \leq s, t \leq 1$$

will shade out the region, just think about the vector addition. The normal will be  $\vec{A} \times \vec{B}$

$$\vec{N}(s, t) = \frac{\partial \vec{\Sigma}}{\partial s} \times \frac{\partial \vec{\Sigma}}{\partial t} = \vec{A} \times \vec{B}.$$

this choice of parametrization gives the parallelogram  $P$  a normal pointing into the page.

Def<sup>n</sup>/ Suppose  $\vec{x}: D \rightarrow S$  where  $(u, v) \mapsto \vec{x}(u, v)$ ,  $f$  continuous on  $S$ ,

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{x}(u, v)) |\vec{x}_u \times \vec{x}_v| dA$$

- we may refer to this as a scalar surface integral. It takes a weighted sum of the scalar function  $f$  over the surface.

Def<sup>b</sup>/ The scalar surface integral of  $f = 1$  over  $S$  gives the surface area

E154 Consider the sphere. Let's find its surface area. Notice

$$|\vec{N}(\varphi, \theta)| = |\vec{x}_\varphi \times \vec{x}_\theta| = |R^2 \sin \varphi e_\rho| = R^2 \sin \varphi \quad \text{since } |e_\rho| = 1.$$

$$\begin{aligned} A(S^2) &= \iint_{S^2} 1 dS = \iint_0^{2\pi} \int_0^\pi R^2 \sin \varphi d\varphi d\theta \\ &= \int_0^{2\pi} R^2 (-\cos \varphi \Big|_0^\pi) d\theta, \quad -\cos \varphi \Big|_0^\pi = -\cos \pi + \cos(0) \\ &= \int_0^{2\pi} 2R^2 d\theta \\ &= \boxed{4\pi R^2} \end{aligned}$$

E155 If  $f(x, y, z) = \sigma(x, y, z)$  = a mass density then the scalar surface integral calculates the total mass of the surface  $S$ . Suppose that  $\sigma(x, y, z) = z^2$ . Again consider the sphere  $S$ ,

$$\begin{aligned} \iint_S \sigma(x, y, z) dS &= \iint_0^{2\pi} \int_0^\pi z^2 R^2 \sin \varphi d\varphi d\theta \quad \text{but } z = R \cos \varphi \\ &= \int_0^{2\pi} \int_0^\pi R^4 \cos^2 \varphi \sin \varphi d\varphi d\theta \\ &= (2\pi R^4) \left( -\frac{1}{3} \cos^3 \varphi \Big|_0^\pi \right) \\ &= (2\pi R^4) \left( -\frac{1}{3} \cos^3 \pi + \frac{1}{3} \cos^3 0 \right) \\ &= \boxed{\frac{4\pi R^4}{3}} = \text{mass of the surface } S. \end{aligned}$$

E156 The graph  $z = f(x, y)$  can be cast as a parametric surface with parameters  $x \neq y$ . Let  $(x, y) \in \text{dom}(f)$  then

$$\vec{x}(x, y) = (x, y, f(x, y))$$

$$\frac{\partial \vec{x}}{\partial x} = \langle 1, 0, \frac{\partial f}{\partial x} \rangle$$

$$\frac{\partial \vec{x}}{\partial y} = \langle 0, 1, \frac{\partial f}{\partial y} \rangle$$

$$\vec{N}(x, y) = \left( \hat{i} + \frac{\partial f}{\partial x} \hat{h} \right) \times \left( \hat{j} + \frac{\partial f}{\partial y} \hat{h} \right) = \frac{\partial f}{\partial y} \hat{i} \times \hat{h} + \frac{\partial f}{\partial x} \hat{h} \times \hat{j} + \hat{i} \times \hat{j}$$

$$\therefore \vec{N}(x, y) = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle$$

$$\Rightarrow \text{Area of graph } f = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} dA$$

Consider  $z = x^2 + y^2$  with  $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$  find the surface area of this paraboloid.

$$\begin{aligned} \text{Area} &= \iint_D \sqrt{(2x)^2 + (2y)^2 + 1} dA && : \text{use polar coordinates to} \\ &= \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} r dr d\theta && \text{integrate, } x = r\cos\theta, y = r\sin\theta \\ &= (2\pi) \frac{2}{3} \frac{1}{8} (4r^2 + 1)^{3/2} \Big|_0^3 && 4r^2 + 1 = r^2 + r^2 + 1 = r^2 + y^2 + x^2 \\ &= \boxed{\frac{\pi}{6} [(37)^{3/2} - 1]} \end{aligned}$$

E157 Some surfaces are not smooth everywhere, but as long as they're piecewise smooth we can integrate. The unit cube for example.  $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

$$\vec{\Sigma}_1(y, x) = (x, y, 0) \Rightarrow \vec{N}_1 = \langle 0, 0, -1 \rangle$$

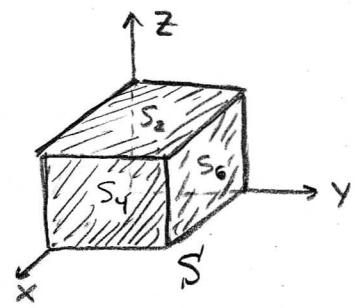
$$\vec{\Sigma}_2(x, y) = (x, y, 1) \Rightarrow \vec{N}_2 = \langle 0, 0, 1 \rangle$$

$$\vec{\Sigma}_3(z, y) = (0, y, z) \Rightarrow \vec{N}_3 = \langle -1, 0, 0 \rangle$$

$$\vec{\Sigma}_4(y, z) = (1, y, z) \Rightarrow \vec{N}_4 = \langle 1, 0, 0 \rangle$$

$$\vec{\Sigma}_5(x, z) = (x, 0, z) \Rightarrow \vec{N}_5 = \langle 0, -1, 0 \rangle$$

$$\vec{\Sigma}_6(z, x) = (x, 1, z) \Rightarrow \vec{N}_6 = \langle 0, 1, 0 \rangle$$



I've given  $S$  a positive orientation. Calculate  $\iint_S xyz dS$

$$\begin{aligned} \iint_S x dS &= \iint_{S_1} xy dS + \iint_{S_2} xz dS + \iint_{S_3} x yz dS + \iint_{S_4} x yz dS + \iint_{S_5} x yz dS + \iint_{S_6} x yz dS \\ &= \iint_{S_2} x yz dS + \iint_{S_4} x yz dS + \iint_{S_6} x yz dS \end{aligned}$$

$$= \iint_0^1 xy(1) dx dy + \int_0^1 \int_0^1 (1)yz dy dz + \int_0^1 \int_0^1 x(1)z dx dz$$

$$= \left( \frac{1}{2}x^2 \Big|_0^1 \right) \left( \frac{1}{2}y^2 \Big|_0^1 \right) + \frac{1}{4} + \frac{1}{4}$$

$$= \boxed{\frac{3}{4}}$$

Remark: Example 3 on p. 1120 is worth looking over.