

SURFACE INTEGRALS OF VECTOR FIELDS

407

We define $\hat{n} = \frac{\vec{N}}{|\vec{N}|}$ to be the unit normal to the parametric surface. We know that $\vec{F} \cdot \hat{n}$ will measure the component of \vec{F} which points along the normal to the surface,

Defⁿ If \vec{F} is a continuous vector field defined on an oriented surface with unit normal \hat{n} , then the surface integral or flux of \vec{F} over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS$$

we may also employ the notation $\iint_S \vec{F} \cdot d\vec{A}$. When S is piecewise smooth we take the sum of the pieces.

Suppose we have the parametrization $\vec{x}(u, v)$ then

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \hat{n} dS \\ &= \iint_D \vec{F}(\vec{x}(u, v)) \cdot \left(\frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} \right) |\vec{x}_u \times \vec{x}_v| du dv \\ &= \iint_D \vec{F}(\vec{x}(u, v)) \cdot (\vec{x}_u \times \vec{x}_v) du dv = \iint_S \vec{F} \cdot d\vec{S} \quad (*) \end{aligned}$$

E158 Suppose $\vec{J} = \frac{\text{current}}{\text{area}}$ then $\vec{J} \cdot d\vec{A}$ will give us the current that cuts through the area perpendicularly. In particular suppose $\vec{J} = r \hat{k}$ for $0 \leq r \leq a$ and $\vec{J} = 0$ for $r \geq a$ find the current through the xy -plane. (call it S)

$$\begin{aligned} I &= \iint_S \vec{J} \cdot d\vec{A} = \int_0^{2\pi} \int_0^a (r \hat{k}) \cdot (r dr d\theta \hat{k}) : \text{choosing the upward orientation for } xy\text{-plane} \\ &= \int_0^{2\pi} d\theta \int_0^a r^2 dr \\ &= 2\pi a^3 / 3 = I_{\text{enclosed by } xy\text{-plane}} = I_{\text{enc.}} \end{aligned}$$

Challenge: find I_{enc} by $ax + by + cz = 0$, same current density.

Remark: in [E158] I pretty much ignored the def² and applied what I would call the geometrical approach. It does match up with the sol² written straight from the def² lets see how,

[E159] Again suppose $\vec{J} = \langle 0, 0, r \rangle$ for $r \leq a$ and $\vec{J} = 0$ for $r > a$, we're using cylindrical coordinates.

$$\vec{\Sigma}(r, \theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle, \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

this parametrizes the xy-plane where $\vec{J} \neq 0$, call it \tilde{S}

$$\vec{\Sigma}_r \times \vec{\Sigma}_\theta = \langle \cos\theta, \sin\theta, 0 \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle 0, 0, r\cos^2\theta + r\sin^2\theta \rangle$$

$$= \langle 0, 0, r \rangle = r\hat{k}$$

Then apply the definition to calculate $\iint_{\tilde{S}} \vec{J} \cdot d\vec{A}$ (\tilde{S} is the disk of radius a in xy-plane.)

$$I = \iint_{\tilde{S}} \langle 0, 0, r \rangle \cdot d\vec{S}$$

$$= \int_0^{2\pi} \int_0^a \langle 0, 0, r \rangle \cdot \langle 0, 0, r \rangle dr d\theta : \text{using (*) on 407}$$

$$= \int_0^{2\pi} \int_0^a r^2 dr d\theta$$

$$= 2\pi a^3 / 3.$$

Comment: The infinitesimal vector area element of the surface is what I called $d\vec{A}$. We observe

$$d\vec{A} = (\vec{\Sigma}_u \times \vec{\Sigma}_v) du dv$$

Then you can see my "geometrical approach" is just another way of assembling the surface integral.

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F} \cdot (\vec{\Sigma}_u \times \vec{\Sigma}_v) du dv = \iint_S \vec{F} \cdot d\vec{S}$$

E160 Let $\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{e}_p}{r^2}$ where Q, ϵ_0 are constants.

Find the flux of \vec{E} through a sphere of radius R called S .

From E152 we recall $\vec{x}(\varphi, \theta) = (R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi)$

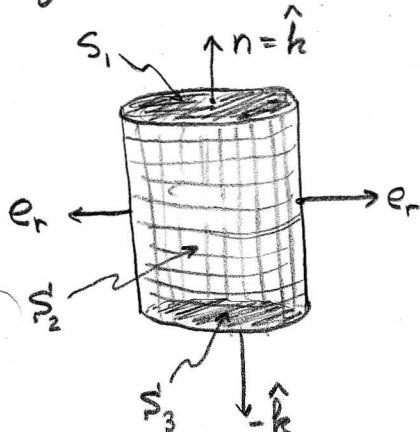
parametrizes the sphere where $0 \leq \varphi \leq \pi$ & $0 \leq \theta \leq 2\pi$ and we calculated that $\vec{x}_\varphi \times \vec{x}_\theta = R^2 \sin\varphi \vec{e}_p$ we find that the vector area element to the sphere is $d\vec{A} = (R^2 \sin\varphi d\varphi d\theta) \vec{e}_p$

$$\begin{aligned}\Phi_E &= \iint_S \vec{E} \cdot d\vec{A} = \int_0^{2\pi} \int_0^\pi \left(\frac{Q}{4\pi\epsilon_0 R^2} \vec{e}_p \right) \cdot (R^2 \sin\varphi d\varphi d\theta \vec{e}_p) : \text{noting } p=R \\ &= \frac{Q}{4\pi\epsilon_0} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^\pi \sin\varphi d\varphi}_{2} : \vec{e}_p \cdot \vec{e}_p = 1. \\ &= \frac{Q}{4\pi\epsilon_0} \cdot 4\pi \\ &= \boxed{Q/\epsilon_0 = \Phi_E} \quad \text{this is an example of the integral form of Gauss' Law.}\end{aligned}$$

Remark: Spheres come up often. Its worth noting that

$d\vec{A} = (R^2 \sin\varphi d\varphi d\theta) \vec{e}_p$. Also you should be aware of the concept of "solid angle", its denoted Ω and $d\Omega = \sin\varphi d\varphi d\theta$. The Ω_{total} for a sphere is 4π .

E161 Let $\vec{B} = \frac{\mu_0 I}{2\pi r} \vec{e}_\theta$ find the flux of \vec{B} through a cylinder of radius a and height h with base on xy -plane



$$\begin{aligned}d\vec{A}_1 &= (r dr d\theta) \hat{k} \\ d\vec{A}_2 &= (r d\theta dz) \vec{e}_r \\ d\vec{A}_3 &= (r dr d\theta) (-\hat{k})\end{aligned} \quad \left. \right\} \text{all orthogonal to } \vec{e}_\theta$$

$$\therefore \boxed{\vec{B} \cdot d\vec{A} = 0}$$

$$\boxed{\Phi_B = \iint_S \vec{B} \cdot d\vec{A} = 0}$$

this is an example of the integral form of the no magnetic monopoles eqⁿ
 $\nabla \cdot \vec{B} = 0$, we'll see why soon.

E162 Let $\vec{F} = \langle P, Q, R \rangle$ and suppose for $(x, y) \in D$
 the surface S is a graph $z = g(x, y)$. In parametrized form

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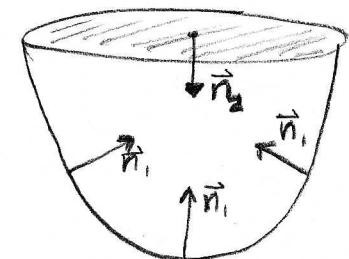
$$\vec{\Sigma}(x, y) = (x, y, g(x, y))$$

$$\vec{\Sigma}_x \times \vec{\Sigma}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle dA$$

Consider then the case $\vec{F} = \langle 0, y, -z \rangle$ and the surface S consists of the paraboloid $z = x^2 + y^2$ and the disk at $z = 1$ which caps the paraboloid. Orient S inwards, call the disk S_2' and the paraboloid S_1' , so $S' = S_1' \cup S_2'$.

$$\begin{aligned} \iint_{S_1'} \vec{F} \cdot d\vec{S} &= \iint_D \langle 0, y, -z \rangle \cdot \langle -2x, -2y, 1 \rangle dA \\ &= \iint_D (-2y^2 - z) dA \quad D = \{(x, y) \mid x^2 + y^2 \leq 1\} \\ &\quad z = x^2 + y^2 = r^2 \\ &= \int_0^{2\pi} \int_0^1 (-2r^2 \sin^2 \theta - r^2) r dr d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{2} r^4 \sin^2 \theta - \frac{1}{4} r^4 \right]_0^1 d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{2} \sin^2 \theta - \frac{1}{4} \right) d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{4} (1 - \cos(2\theta)) - \frac{1}{4} \right) d\theta \\ &= \left(-\frac{1}{2}\theta + \frac{1}{8}\sin(2\theta) \right]_0^{2\pi} \\ &= -\pi \end{aligned}$$



$$\iint_{S_2'} \vec{F} \cdot d\vec{S} = \iint_D \langle 0, y, -1 \rangle \cdot \langle 0, 0, -1 \rangle dA \quad (\text{on } S_2' z = 1.)$$

$$= \int_0^{2\pi} \int_0^1 r dr d\theta$$

$$= \boxed{\pi}$$

$$\therefore \boxed{\iint_S \vec{F} \cdot d\vec{S} = \pi - \pi = 0.}$$

E163 Let \vec{F} be an inverse square field, use physics notation,

$\vec{F} = \frac{c}{r^3} \hat{r}$ where $\hat{r} = \langle x, y, z \rangle$. Show that the flux of \vec{F} across a sphere centered at the origin is independent of the radius R of the sphere.

$$\begin{aligned}
 \Phi_F &= \iint_{S_R} \vec{F} \cdot d\vec{A} \\
 &= \iint_{S_R} \left(\frac{c}{R^2} \hat{r} \right) \cdot \left(R^2 \sin \theta d\theta d\phi \hat{r} \right) & : 0 \leq \theta \leq \pi \quad \text{and } r=R \\
 &= \int_0^{2\pi} \int_0^\pi c \sin \theta d\theta d\phi \\
 &= c \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \\
 &= \boxed{4\pi c} = \Phi_F
 \end{aligned}$$

the flux is independent of the radius for this special type of field.