

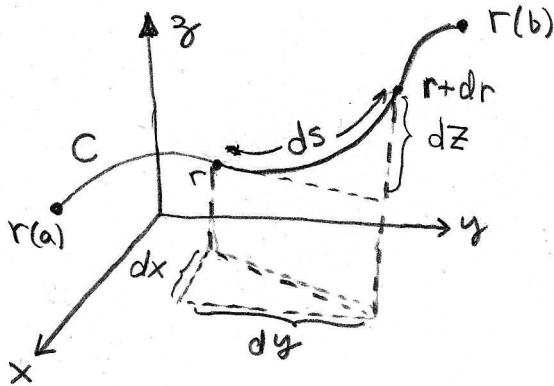
We study a path $\vec{r}: [a, b] \rightarrow \mathbb{R}^3$. The point set $\vec{r}[a, b] \subset \mathbb{R}^3$ is called a curve. There are many paths corresponding to the same curve, however much about the curve can be calculated from an arbitrary path. These properties of the curve are called intrinsic. Essentially the curvature (K), torsion (τ) and arclength (s) characterize curves upto a translation (I'll give careful Th later). The curvature and torsion arise in the study of the moving frame $\vec{T}, \vec{N}, \vec{B}$ as we shall see. To begin we study arclength.

ARCLENGTH

Imagine a curve C in \mathbb{R}^3 , place a length ds of string along the curve then by the distance formula,

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

where dx, dy, dz are the displacements in x, y, z necessary to reach from \vec{r} to $\vec{r} + d\vec{r}$ where $\vec{r} + d\vec{r}$ is the place on the curve reached by going ds from \vec{r} .



Now parametrize the curve by $\vec{r}: [a, b] \rightarrow C \subset \mathbb{R}^3$. We have that $dx = \frac{dx}{dt} dt$, $dy = \frac{dy}{dt} dt$, $dz = \frac{dz}{dt} dt$ for the infinitesimal arc considered, dt = time to traverse ds along C .

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \Rightarrow ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

$$\therefore s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

arclength of C

Remark: technically we mean to approx. the path by piecewise polygonal path then take limiting process. The infinitesimals are simply an abbreviation for that.

ARCLENGTH CONTINUED

The path $\vec{r}: [a, b] \rightarrow C \subset \mathbb{R}^3$ with parameter t has

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$|\vec{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Thus the arclength can be written as for C from $\vec{r}(a)$ to $\vec{r}(b)$,

$$S = \int_a^b |\vec{r}'(t)| dt$$

E32 Consider $\vec{r}(t) = \langle R \cos t, R \sin t \rangle$ for $R > 0$. Notice that $\vec{r}'(t) = \langle -R \sin t, R \cos t \rangle$ thus $|\vec{r}'(t)| = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R$.

We want a circle so $0 \leq t \leq 2\pi$ then $a=0, b=2\pi$ so

$$S = \int_0^{2\pi} R dt = R t \Big|_0^{2\pi} = 2\pi R$$

the arclength of a circle is $2\pi R$, otherwise we'd be in trouble!

E33 A helix with slope b is given by $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$ for $0 \leq t \leq 2\pi$ (could let t keep going if you want the helix to continue onward.) anyway calculate, we assume a, b are constants.

$$\vec{r}'(t) = \langle -a \sin t, a \cos t, b \rangle$$

$$|\vec{r}'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\therefore S = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = (\sqrt{a^2 + b^2}) t \Big|_0^{2\pi} = 2\pi \sqrt{a^2 + b^2}$$

Of course when $b=0$ we get a circle and we recover $2\pi a$ in that case (a good check of things here.)

Remark: to calculate arclength of C we need the curve to be smooth, well at least once differentiable. If the curve has finitely many kinks in it then we can find the total length by adding the differentiable segments lengths together.

ARCLENGTH AS A PARAMETER & REPARAMETRIZATION

Suppose we have a curve C with a nonstop parametrization
 $\vec{r}: [a, b] \rightarrow C \subset \mathbb{R}^3$. (nonstop means $\vec{r}'(t) \neq 0 \quad \forall t \in [a, b]$)
 then the arclength from

$\vec{r}(a)$ upto $\vec{r}(t)$ is itself a function of t given by

$$s(t) = \int_a^t |\vec{r}'(\tau)| d\tau$$

τ = dummy variable
 of integration could use
 t', \bar{t}, u \oplus whatever
 but not t .

Consider then, by the fundamental Th² of calculus,

$$(*) \quad \frac{ds}{dt} = \frac{d}{dt} \int_a^t |\vec{r}'(\tau)| d\tau = |\vec{r}'(t)| = \text{Speed of } \vec{r}(t) \text{ at time } t.$$

By assumption $|\vec{r}'(t)| \neq 0 \quad \forall t \Rightarrow \frac{ds}{dt} > 0 \quad \forall t$

thus s is a strictly increasing function of $t \Rightarrow s(t)$

can be inverted to give $t(s)$. Thus we can then

write $\vec{r}(s) \equiv \vec{r}(t(s))$, replace t with t in terms of s .

E34 The helix $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle \quad 0 \leq t \leq 2\pi$

we found $|\vec{r}'(t)| = \sqrt{a^2+b^2}$ thus,

$$s(t) = \int_0^t \sqrt{a^2+b^2} d\tau = (\sqrt{a^2+b^2})\tau \Big|_0^t = t\sqrt{a^2+b^2}$$

$$\Rightarrow t = \frac{s(t)}{\sqrt{a^2+b^2}} \quad \text{or changing notation, } t(s) = \frac{s}{\sqrt{a^2+b^2}}$$

- it may be wise to suppress the of (t) and of (s) dependence here. Be careful not to confuse $s(t)$ with s times. We find that

$$\vec{r}(t(s)) = \boxed{\vec{r}(s) = \langle a \cos \left(\frac{s}{\sqrt{a^2+b^2}} \right), a \sin \left(\frac{s}{\sqrt{a^2+b^2}} \right), b \frac{s}{\sqrt{a^2+b^2}}} \rightarrow$$

Remark: this notation that $\vec{r}(t)$ and $\vec{r}(s)$ are different is ambiguous (but popular). Technically we ought to use a different symbol for $\vec{r}(s)$, like $\vec{r}_s(s)$ where $\vec{r}_s(s) = \vec{r}(t(s))$. Otherwise, what is meant by $\vec{r}(1)$? Is that $\vec{r}(t=1)$ or $\vec{r}(s=1)$?

E35 Let $\vec{r}(t) = \langle t, \frac{\sqrt{t}}{2}t^2, \frac{1}{3}t^3 \rangle$, $0 \leq t \leq 1$. Find the arclength function.

$$\vec{r}'(t) = \langle 1, \sqrt{t}, t^2 \rangle \Rightarrow |\vec{r}'(t)| = \sqrt{1 + 2t^2 + t^4} = \sqrt{(1+t^2)^2}$$

Thus,

$$s = \int_0^t \sqrt{(1+\tau^2)^2} d\tau = \int_0^t (1+\tau^2) d\tau = \left[\tau + \frac{1}{3}\tau^3 \right]_0^t = \boxed{t + \frac{1}{3}t^3 = s}$$

I would like to solve this for t , however it seems out of reach, at least in terms of elementary functions. Bonus point if you can find for me $f(s)$ which is a power series for which $f(s) + \frac{1}{3}(f(s))^3 = s$ that is a power series sol^o for t in terms of just s .

Remark: $s(t)$ may be impossible to calculate in a nice closed form and even when it is $t(s)$ may be very subtle to calculate. Even so the arclength plays an important allbeit implicit role.

Proposition: A path $\vec{r}: [0, b] \rightarrow C$ which has arclength as its parameter has unit-speed $\forall s$.

Proof: unit speed means $|\vec{r}'(s)| = 1$. Consider the chain-rule,

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \frac{ds}{dt} = \frac{d\vec{r}}{ds} |\vec{r}'(t)| \quad \text{using } (*) \text{ on 271}$$

$$\Rightarrow \frac{d\vec{r}}{ds} = \frac{1}{|\vec{r}'(t)|} \frac{d\vec{r}}{dt} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad (\text{recall we assume } \vec{r}(t) \text{ is nonstop.})$$

$$\Rightarrow \left| \frac{d\vec{r}}{ds} \right| = \left| \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \right| = \frac{|\vec{r}'(t)|}{|\vec{r}'(t)|} = 1 //.$$

Remark: the "speed" of a curve is the change in arclength with respect to the change in parameter so the change in arclength w.r.t. arclength is one.

CURVATURE AND THE UNIT TANGENT VECTOR \vec{T}

to begin we define the unit tangent vector to be the tangent vector normalized to length one. Assume $\vec{r}: [a, b] \rightarrow C \subset \mathbb{R}^3$ is a nonstop ($\vec{r}'(t) \neq 0 \quad \forall t \in [a, b]$) path

$$\text{Defn} / \quad \vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{|\vec{r}'(t)|} \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \quad \text{for arclength } s$$

E36 $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle, \quad 0 \leq t \leq 2\pi$: the helix,

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{a^2+b^2}} \langle -a \sin t, a \cos t, b \rangle = \vec{T}(t)$$

Using $|\vec{r}'(t)| = \sqrt{a^2+b^2}$ as we found previously in E33, alternatively recall from E34 that we found C parametrized by arclength,

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right\rangle$$

$$\vec{T}(s) = \frac{d\vec{r}}{ds} = \frac{1}{\sqrt{a^2+b^2}} \left\langle -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), b \right\rangle \quad \leftarrow$$

consistent with $\vec{T}(t)$ found above
since from E34 $t = \frac{s}{\sqrt{a^2+b^2}}$.

I factored
out the
 $\frac{1}{\sqrt{a^2+b^2}}$
from the
chainrules.

PROPOSITION: Assume that $\vec{r}: [a, b] \rightarrow C \subset \mathbb{R}^3$ is nonstop then

(i.) $\vec{T}'(t)$ is orthogonal to $\vec{T}(t) \quad \forall t \in [a, b]$.

(ii.) $\left| \frac{d\vec{T}}{dt} \Big|_{t=t_0} \right|$ = angular rate of change with respect to t
of the direction of \vec{T} when $t = t_0$.

Proof: the proof of (i.) is easy, (ii) requires a little thought.

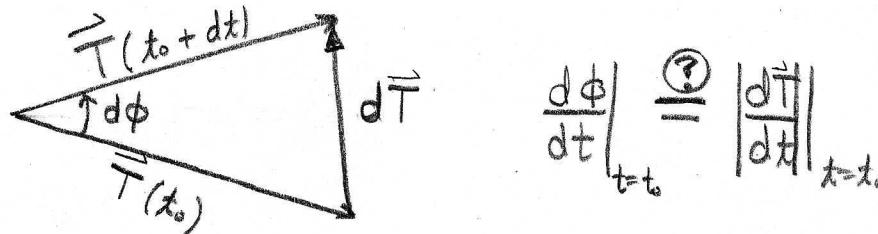
(i.) $\vec{T} \cdot \vec{T} = |\vec{T}|^2 = 1$ then differentiate and use property

(iv) from 265 to obtain,

$$\frac{d}{dt} (1) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T} \cdot \frac{d\vec{T}}{dt} \Rightarrow 0 = 2\vec{T} \cdot \frac{d\vec{T}}{dt} \therefore \underline{\underline{\vec{T} \cdot \vec{T}' = 0}} //$$

Proof continued:

For (ii.) consider the infinitesimal argument below,



$$\left| \frac{d\phi}{dt} \right|_{t=t_0} \stackrel{?}{=} \left| \frac{dT}{dt} \right|_{t=t_0}$$

this is what we seek to show.

Notice that the law of Cosines yields, use $|T| = 1$ to simplify,

$$|dT|^2 = |\vec{T}(t_0 + dt)|^2 + |\vec{T}(t_0)|^2 - 2|\vec{T}(t_0 + dt)||\vec{T}(t_0)| \cos(d\phi)$$

$$|dT|^2 = 2 - 2\cos(d\phi) = 2 - 2(1 - \frac{1}{2}(d\phi)^2 + \dots)$$

$$|dT|^2 = (d\phi)^2 \Rightarrow |dT| = d\phi \Rightarrow \left| \frac{dT}{dt} \right|_{t=t_0} = \left| \frac{d\phi}{dt} \right|_{t=t_0} . //$$

Remark: See Colley's Text for a more conventional proof if you wish.

Def^e/ The curvature κ of a path $\vec{r}: [a, b] \rightarrow C \subset \mathbb{R}^3$ is the angular rate of change of the direction of \vec{T} per unit change in the distance along the path.

this def^e is quite geometric, we use the Prop. on (273) plus our knowledge about arclength to obtain a more practical formulation

$$\kappa(t) \equiv \frac{\frac{d\phi}{dt}}{\frac{ds}{dt}} \stackrel{\substack{\text{Prop.} \\ (273)}}{=} \frac{|dT/dt|}{ds/dt} = \left| \frac{dT}{ds} \right|$$

Since for arclength parametrization $ds/ds = 1$. Thus we may use,

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{T}'(t)|} \quad | \vec{T}'(s) | = \kappa(s)$$

Remark: #52 of §14.3 in Stewart is intended to make more explicit the $d\phi/ds$ idea. You might notice Stewart only says in words what we have derived to prove the prop on (273). I'm following Colley's §3.2 pretty closely here. She has a bit more detail than Stewart offers.

E37 The circle $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$, $0 \leq t \leq 2\pi$, assume $a > 0$,

$$\vec{r}'(t) = \langle -a \sin t, a \cos t \rangle \quad \therefore |\vec{r}'(t)| = \frac{ds}{dt} = \sqrt{a^2} = a$$

Thus the unit tangent,

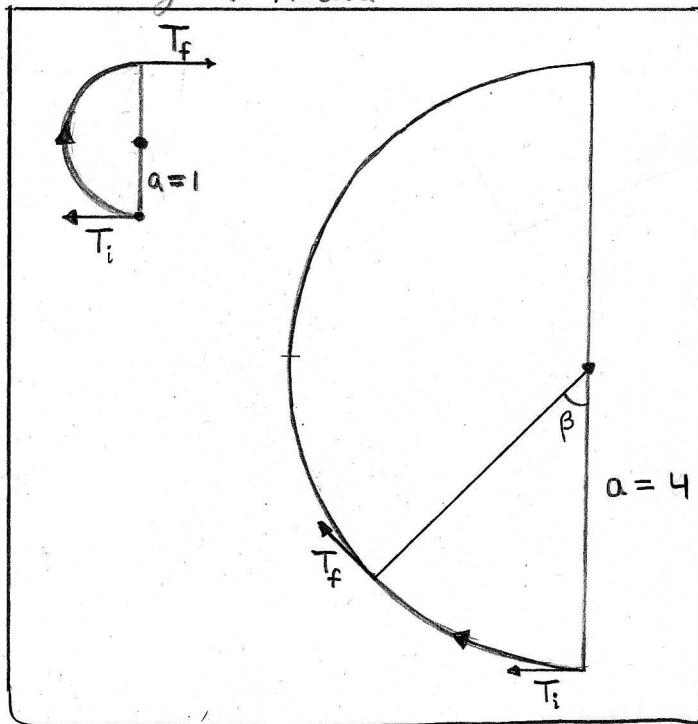
$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle -\sin t, \cos t \rangle$$

$$\vec{T}'(t) = \langle -\cos t, -\sin t \rangle \quad \therefore |\vec{T}'(t)| = 1.$$

$$\therefore \kappa(t) = \frac{|\vec{T}'(t)|}{ds/dt} = \boxed{\frac{1}{a}} = \kappa$$

: the curvature of a circle is inversely proportional to the radius of the circle.

I'll endeavor to sketch this, it's clear that smaller circles force the \vec{T} -vector to turn all the way around quicker for a given arc length. Recall $s = a\theta$ need $4\beta = \pi \Rightarrow \beta = \pi/4$ on big circle gives same arc length.



You can clearly see that the Tangent vector completely reverses direction for the circle of $R = 1$. whereas for the larger circle of $R = 4$ the Tangent vector only changes direction by $\pi/4$ relative to its initial state.

E38 Consider the line $\vec{r}(t) = \vec{a}t + \vec{b}$ for \vec{a}, \vec{b} fixed vectors. Then $\vec{r}'(t) = \vec{a}$ thus $\vec{T}(t) = \frac{\vec{a}}{|\vec{a}|}$ hence $\vec{T}'(t) = 0$. Consequently

$$\kappa(t) = \frac{|\vec{T}'(t)|}{ds/dt} = \boxed{0 = \kappa}$$

lines have no curvature. One might also anticipate this result from **E37** taking the radius $a \rightarrow \infty \Rightarrow \frac{1}{a} \rightarrow 0$.

E39 The helix, $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$. Recall we found
in E36 that

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$$\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a \sin t, a \cos t, b \rangle \quad \text{and} \quad |\vec{r}'(t)| = \frac{ds}{dt} = \sqrt{a^2+b^2}$$

Calculate then,

$$\vec{T}'(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a \cos t, -a \sin t, 0 \rangle$$

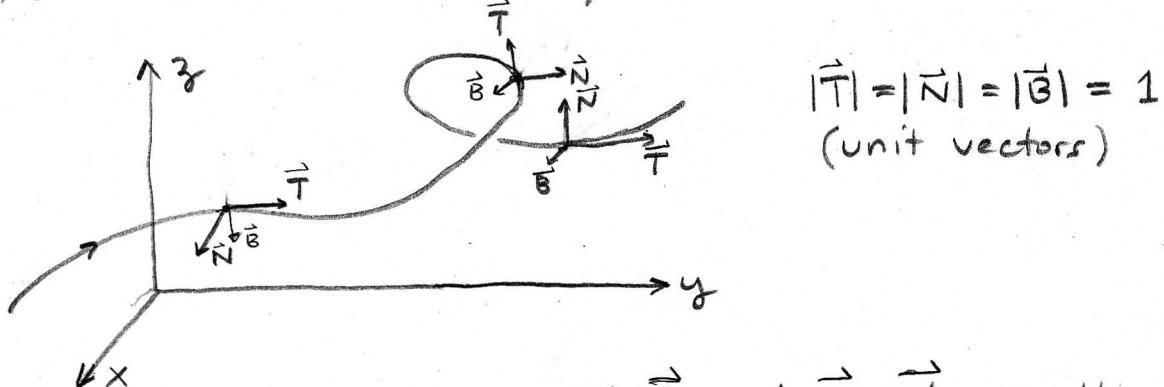
$$|\vec{T}'(t)| = \frac{1}{\sqrt{a^2+b^2}} \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\therefore \kappa(t) = \frac{|\vec{T}'(t)|}{ds/dt} = \frac{a/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \boxed{\frac{a}{a^2+b^2} = \kappa}$$

Again when $b=0$ we find $\kappa = \frac{a}{a^2} = \frac{1}{a}$ which is consistent with what we found for the circle in E37.

$\{\vec{T}, \vec{N}, \vec{B}\}$ & THE OSCULATING PLANE

Suppose that we have a path \vec{r} such that $\vec{r}'(t) \neq 0$ and $\vec{r}'(t) \times \vec{r}''(t) \neq 0 \ \forall t$. (it's nonstop & not a line) then we can define three vectors which provide an orthogonal basis that moves with the path.



The osculating plane has normal \vec{B} and \vec{T}, \vec{N} actually reside in this plane. Moreover "locally" the osculating plane contains the path. Define,

$\vec{T}(t) = \frac{\vec{r}'(t)}{ \vec{r}'(t) }$	$\vec{N}(t) = \frac{\vec{T}'(t)}{ \vec{T}'(t) }$	$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
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We proved $\vec{T} \cdot \vec{N} = 0$ essentially back in prop (i) on 273. The fact that $\vec{N} \cdot \vec{B} = 0$ and $\vec{T} \cdot \vec{B} = 0$ follows immediately from the def² of \vec{B} .

Proposition: $\frac{d\vec{T}}{ds} = \kappa \vec{N}$ for a nonlinear nonstop path.

Proof: not much to say here. Just recall definitions, and (274),

$$\vec{N} = \frac{\vec{T}'(s)}{|\vec{T}'(s)|} \quad \text{and} \quad \kappa = \left| \frac{d\vec{T}}{ds} \right| = |\vec{T}'(s)|$$

$$\therefore \vec{N} = \frac{\vec{T}'(s)}{\kappa} \Rightarrow \vec{T}'(s) = \boxed{\frac{d\vec{T}}{ds} = \kappa \vec{N}} //$$

Notice we must avoid lines since $\kappa=0$ for lines!

Proposition: $\frac{d\vec{B}}{ds} \cdot \vec{T} = 0$ so $\frac{d\vec{B}}{ds} = -\tau \vec{N}$ for some scalar function τ

Proof: see hwh sol¹² §10.3 #47. I rehash much of the material here and give an explicit careful proof of this Prop. // §10.3 #47, 1 → §14.3 #53 in Stewart's Calculus version 6

We now may be certain the defⁿ below is logical,

Defⁿ / The torsion $\tau(s)$ is the function of arclength such that

$$\frac{d\vec{B}}{ds} = -\tau \vec{N}$$

geometrically the torsion measures how the curve twists out of the osculating plane. Thus for a curve which lies in a plane one finds the torsion is zero. Before any examples lets complete the theory, the following formulas were discovered about 1850 by Frédéric-Jean Frenet and Joseph Alfred Serret. They're known as the Frenet-Serret formulas. (See hwhs # 47 & 48 of §10.3 for proof.)

$\frac{d\vec{T}}{ds} = \kappa \vec{N}$	$\frac{d\vec{N}}{ds} = -\kappa \vec{T} + \tau \vec{B}$	$\frac{d\vec{B}}{ds} = -\tau \vec{N}$
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Finally I quote a Th^m from Colley (2.5) which goes to show just how fundamental the quantities κ and τ are to describing a curve.

Th^m / Let C_1 and C_2 be smooth curves in \mathbb{R}^3 both with strictly positive curvatures κ_1 and κ_2 . Then if $\kappa_1(s) = \kappa_2(s)$ and $\tau_1(s) = \tau_2(s)$. As then the curves C_1 & C_2 are congruent in the sense of highschool geometry ($C_1 = C_2 + \vec{b}$ some fixed vector). Moreover the converse is true, given a positive arclength function and torsion one can uniquely reconstruct a curve upto translations.

E40 Consider the circle, $\vec{r}(t) = \langle a \cos t, a \sin t, 0 \rangle$. We found in **E37** that $\vec{T}(t) = \langle -\sin t, \cos t, 0 \rangle$ and $\vec{T}'(t) = \langle -\cos t, -\sin t, 0 \rangle$ so $|\vec{T}'(t)| = 1$ thus we find $\vec{N}(t) = \vec{T}'(t) = \langle -\cos t, -\sin t, 0 \rangle$. Notice that $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, \sin^2 t + \cos^2 t \rangle = \langle 0, 0, 1 \rangle$.

Thus \vec{B} is a constant vector which means we get $\vec{B}(s) = \langle 0, 0, 1 \rangle$, $\frac{d\vec{B}}{ds} = 0 = -\tau \vec{N} \Rightarrow \tau = 0$

This is good, we predicted that planar curves (like a circle) have zero torsion. We should note that usually we will need to reparametrize $\vec{B}(t)$ to $\vec{B}(s)$ to find τ , this case was special.

E41 The helix $\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle$ (see **E33**, **E34**, **E36**, **E39**) to find $\vec{T}(t)$ and $\vec{T}'(t)$

$$\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a \sin t, a \cos t, b \rangle$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \left(\frac{\sqrt{a^2+b^2}}{a} \right) \frac{1}{\sqrt{a^2+b^2}} \langle -a \cos t, -a \sin t, 0 \rangle = \langle -\cos t, -\sin t, 0 \rangle. \quad (***)$$

$$\vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin t & a \cos t & ab \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

I let $\alpha \equiv \frac{1}{\sqrt{a^2+b^2}}$ because I'm tired of $\sqrt{a^2+b^2}$

$$= \langle \alpha b \sin t, -\alpha b \cos t, \alpha a \sin^2 t + \alpha a \cos^2 t \rangle$$

$$= \alpha \cdot \langle b \sin t, -b \cos t, a \rangle = \vec{B}(t)$$

We found in **E34** on 271 that $t = s/\sqrt{a^2+b^2} = s\alpha$ thus,

$$\vec{B}(s) = \alpha \langle b \sin(\alpha s), -b \cos(\alpha s), a \rangle$$

$$\therefore \frac{d\vec{B}}{ds} = \alpha^2 \langle b \cos(\alpha s), b \sin(\alpha s), 0 \rangle$$

$$= -b\alpha^2 \langle -\cos(\alpha s), -\sin(\alpha s), 0 \rangle : \text{this is } (**) \text{ reparam. with } t = \alpha s. \text{ This is } \vec{N}(s).$$

Comparing we find $\tau = b\alpha^2 = \frac{b}{a^2+b^2} = \tau$

Notice that as $b \rightarrow 0$ the torsion goes to zero, which is in agreement with **E40**.

E42 Consider $\vec{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$. Calculate the T, N, B frame and the curvature and torsion. We begin,

(279)

$$\vec{r}'(t) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$$

$$\begin{aligned}\vec{r}'(t) \cdot \vec{r}'(t) &= (e^t)^2 (\cos t - \sin t)^2 + (e^t)^2 (\sin t + \cos t)^2 + (e^t)^2 \\ &= e^{2t} [\cos^2 t - 2\sin t \cos t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t + 1] \\ &= 3e^{2t} \Rightarrow |\vec{r}'(t)| = \sqrt{3} e^t\end{aligned}$$

Thus the unit tangent vector $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$ is

$$\vec{T}(t) = \frac{1}{\sqrt{3}} \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$$

Recall that the chain rule says $\frac{d\vec{T}}{dt} = \frac{ds}{dt} \frac{d\vec{T}}{ds} \therefore \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$
But we know $\frac{ds}{dt} = |\vec{r}'(t)| = \sqrt{3} e^t$ thus we find $\frac{d\vec{T}}{ds}$ as a function of t ,

$$\frac{d\vec{T}}{ds} = \frac{1}{3e^t} \langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle$$

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{3e^t} \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} = \frac{\sqrt{2}}{3e^t} = \kappa(t)$$

I calculate the Normal vector \vec{N} using an indirect method,

$$\vec{N} = \underbrace{\frac{1}{\kappa} \frac{d\vec{T}}{ds}}_{\text{see prop. on (277)}} = \frac{3e^t}{\sqrt{2}} \frac{1}{3e^t} \langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle$$

$$\therefore \vec{N} = \frac{1}{\sqrt{2}} \langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle$$

And now the binormal follows from straightforward computation of $\vec{B} = \vec{T} \times \vec{N}$ which yields, (I leave it for you)

$$\vec{B} = \frac{1}{\sqrt{6}} \langle \sin t - \cos t, -\sin t - \cos t, 2 \rangle$$

Now we may deduce the torsion, again use the chain rule trick,

$$\begin{aligned}\frac{d\vec{B}}{ds} &= \frac{1}{ds/dt} \frac{d\vec{B}}{dt} = \frac{1}{\sqrt{3}e^t} \frac{1}{\sqrt{6}} \langle \cos t + \sin t, -\cos t + \sin t, 0 \rangle \\ &= \underbrace{\left(\frac{-1}{3e^t} \right)}_{-\tau} \underbrace{\frac{1}{\sqrt{6}} \langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle}_{\vec{N}}\end{aligned}$$

$$\tau = \frac{1}{3e^t}$$

Remark: the chain rule has saved us the trouble of computing the arclength!