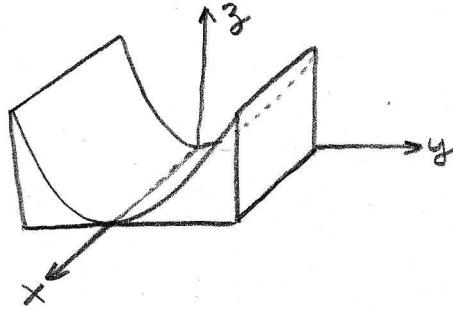


TRIPLE INTEGRALS OVER GENERAL BOUNDED REGIONS IN \mathbb{R}^3

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Rather than explicitly stating the 3-d Fubini Th^m I will simply illustrate with a few examples. Usually we can bound z in terms of $x \& y$ then we can bound y in terms of x or vice-versa, that gives two orders of integration. Then other problems allow x to be bound in terms of $y \& z$ or possibly y in terms of $x \& z$, in total there are 6 ways to write a particular integral over a volume. In §12.7 #31 I explicitly show 6 ways to write a particular integral. I don't give general advice on how to rewrite and switch bounds, it's a subtle business and I would advocate double checking with Mathematica. Generalities aside, let's do a few typical problems.

E100 Let us find the volume of the region between $z = y^2$ and the xy -plane bounded by $x=0$, $x=1$, $y=1$ and $y=-1$. Notice $dV = dx dy dz$ so integrating dV gives volume V ,



$$\begin{aligned}0 &\leq z \leq y^2 \\0 &\leq x \leq 1 \\-1 &\leq y \leq 1\end{aligned}$$

- we must integrate w.r.t. z either first or second.

$$V = \int_{-1}^1 \int_0^1 \int_0^{y^2} dz dx dy : \text{we work inside out as usual.}$$

$$= \int_{-1}^1 \int_0^1 y^2 dx dy : \text{back to 2-d integrals.}$$

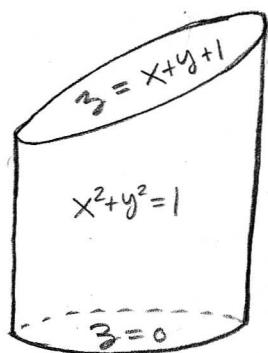
$$= \int_{-1}^1 y^2 dy : \text{back to 1-d integral.}$$

$$= \frac{1}{3} y^3 \Big|_{-1}^1$$

$$= \frac{1}{3} (1 - (-1)^3)$$

$$= \boxed{\frac{2}{3}}$$

E101 Consider the cylinder $x^2 + y^2 = 1$, let $z = 0$ bound it from below and let $z = x + y + 1$ bound it above. Call this solid B. A sketch of G reveals the inequalities to the right of it



$$0 \leq z \leq x + y + 1$$

$$0 \leq x^2 + y^2 \leq 1 \quad \begin{array}{l} \rightarrow -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ \rightarrow -1 \leq x \leq 1 \end{array}$$

Calculate then,

$$\begin{aligned} \iiint_B x \, dV &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{x+y+1} x \, dz \, dy \, dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + xy + x) \, dy \, dx \\ &= \int_{-1}^1 [(x^2 + x)y + \frac{1}{2}xy^2]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx \\ &= \int_{-1}^1 (2x^2\sqrt{1-x^2} + 2x\sqrt{1-x^2}) \, dx \end{aligned}$$

I need to think about this one.

zero, odd function over even
trig substitution.

$$\begin{aligned} \int x^2 \sqrt{1-x^2} \, dx &= \int \sin^2 \theta \cos \theta \cos \theta \, d\theta & \begin{array}{ll} x = \sin \theta & dx = \cos \theta \, d\theta \\ 1-x^2 = \cos^2 \theta & \sqrt{1-x^2} = \cos \theta \end{array} \\ &= \int (\sin^2 \theta - \sin^4 \theta) \, d\theta \\ &= \int [\frac{1}{2}(1-\cos(2\theta)) - \frac{1}{4}(1-2\cos(2\theta)+\cos^2(2\theta))] \, d\theta \\ &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) - \frac{1}{4} + \frac{1}{2}\cos(2\theta) - \frac{1}{4}\cos^2(2\theta)\right) \, d\theta \\ &= \int \left[\frac{1}{4} - \frac{1}{8}(1-\cos(4\theta))\right] \, d\theta \\ &= \theta/8 + \sin(4\theta)/32 \end{aligned}$$

Change bounds on x from $-1=x \rightarrow 1=x \Rightarrow -\frac{\pi}{2}=\theta \rightarrow \frac{\pi}{2}=\theta$.

$$\iiint_B x \, dV = \left[\frac{\theta}{8} + \frac{1}{32}\sin(4\theta) \right]_{-\pi/2}^{\pi/2} = \boxed{\frac{\pi}{4}}$$

E102 Let B be bounded by coordinate planes and the plane passing through $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$. We find the eq² of this plane to begin, note \vec{V} & \vec{W} are on the plane

$$\vec{V} = (0,0,1) - (0,1,0) = \langle 0, -1, 1 \rangle$$

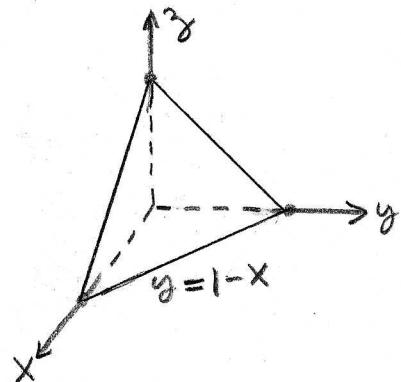
$$\vec{W} = (0,0,1) - (1,0,0) = \langle -1, 0, 1 \rangle$$

$$\vec{V} \times \vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1, -1, -1 \rangle = \langle a, b, c \rangle \text{ the normal.}$$

Choose $r_0 = (0,0,1)$ to base the plane eq²,

$$-x - y - (z - 1) = 0 \Rightarrow z = 1 - x - y$$

Lets plot it, note $z = 1 - x - y$ intersects $z = 0$ on the line $y = 1 - x$ in the xy -plane.



$$\left. \begin{array}{l} 0 \leq z \leq 1 - x - y \\ 0 \leq y \leq 1 - x \\ 0 \leq x \leq 1 \end{array} \right\}$$

this is a useful description of B .

Lets find the volume of B ,

$$\begin{aligned} V &= \iiint_B dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx \\ &= \int_0^1 \int_0^{1-x} (1-x-y) dy dx \\ &= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \Big|_0^{1-x} \right] dx \\ &= \int_0^1 \left[(1-x)^2 - \frac{1}{2}(1-x)^2 \right] dx \\ &= \int_0^1 \frac{1}{2}(1-2x+x^2) dx \\ &= \frac{1}{2} \left(1 - \frac{2}{2} + \frac{1}{3} \right) = \boxed{\frac{1}{6}} \end{aligned}$$

Remark: These problems are pretty much easy once you figure out how to describe the solid.

[E103] find the average value of $f(x, y, z) = x$ on the solid region from [E102]. The average is defined to be

$$\text{f}_{\text{avg}}^{\text{B}} = \frac{1}{\text{Vol}(B)} \iiint_B f(x, y, z) dV$$

We just found $\text{vol}(B) = 1/6$, let's focus on the $\iiint_B f dV$,

$$\begin{aligned} \iiint_B f dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx \\ &= \int_0^1 x \int_0^{1-x} (1-x-y) dy dx \\ &= \int_0^1 x \left[(1-x)y - \frac{1}{2}y^2 \Big|_0^{1-x} \right] dx \\ &= \int_0^1 \frac{1}{2}x(1-x)^2 dx \\ &= \int_0^1 \frac{1}{2}(x - 2x^2 + x^3) dx \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \\ &= \frac{1}{2}\left(\frac{3}{4} - \frac{2}{3}\right) \\ &= \frac{1}{24} \quad \Rightarrow \quad \text{f}_{\text{avg}}^{\text{B}} = \frac{1/24}{1/6} = \boxed{\frac{1}{4}} \end{aligned}$$

that seems pretty reasonable from the picture in [E102].

Remark: We have studied how to integrate in Cartesian coordinates in some detail. It turns out that this is quite limiting. To do many interesting problems with better efficiency it pays to employ cylindrical or spherical coordinates. Before getting to those special choices we consider a general coordinate change briefly and in the process derive what we later use for the cylindrical & spherical coordinates.