VECTOR FIELDS

A vector field is an assigment of a vector at each point in space. Usually we take the vector field to be static, but you could imagine that the assigments itself had a time dependence.

Def^n/ Let D be a subset of $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in D a 2-dim'l vector $\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$.

we call F_1 and F_2 the component functions of $\vec{F}$.

Def^n/ Let $E \subseteq \mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x,y,z) in E a 3-dim'l vector

$$\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$$

The text likes to use the notation P, Q or P, Q, R for the component functions so $\vec{F} = \langle P, Q\rangle$ in 2-dimensions or $\vec{F} = \langle P, Q, R\rangle$ in $\mathbb{R}^3$.

- Each vector field is made of several scalar-valued functions. We could say a vector field is a vector-valued function of $\mathbb{R}^n$. There are many interesting vector fields,
  - $\vec{E}$ the Electric Field
  - $\vec{B}$ the Magnetic Field
  - $\vec{F}$ a force field
  - $\vec{F} = -\frac{GmM}{r^2}\hat{r}$
  - $\vec{V}$ the velocity field of some liquid
  - $\vec{F} = -\nabla U$ a conservative vector field, U is a scalar function we call the potential energy.

Remark: We have already played with vector fields back when we consider the operation of gradients. The object ∇f is a vector field. It turns out that not all vector fields have the form $\vec{F} = \nabla f$. When $\exists f$ such that $\vec{F} = \nabla f$ we say $\vec{F}$ is conservative. (this defines "conservative" for future reference)

E122 Consider $x^2+y^2=1$. Diff. implicitly $\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

in calc II. we plotted $\frac{dy}{dx}$ at each (x,y) to describe the so-called direction-field. This is another important example of vector fields.

WHAT'S THE DEAL WITH E122?

Let me elaborate on what I cut the discussion short on Tuesday October 28, 2008.

Def^n/ $\vec{r}: \mathbb{R} \to \mathbb{R}^n$ is an integral curve (or flow line) of a vector field $\vec{\Sigma}$ on $\mathbb{R}^n$ if we have that the vector field aligns itself with the tangent vectors to the curve, that is to say

$$\vec{\Sigma}(\vec{r}(t)) = \vec{r}\,'(t)$$

Now let's begin with the vector field $\vec{F}(x,y) = \langle -y, x \rangle$ let's suppose $\vec{r}(t) = \langle x(t), y(t) \rangle$ is an integral curve of $\vec{F}$. This means it must be a sol^n to

$$\vec{F}(\vec{r}(t)) = \vec{r}\,'(t)$$

$$\langle -y, x \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

This yields two coupled linear ordinary differential eq^n's. These have a nice sol^n, I'll show it here next,

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x \implies \frac{d^2y}{dt^2} = \frac{dx}{dt} = -y$$

Now the differential eq^n $\boxed{\frac{d^2y}{dt^2} + y = 0}$ (*) has well-known sol^n's of

$$\boxed{y = A\sin(t + \varphi)}$$

where A and φ are constants. You can substitute this into (*) and check it works. Then

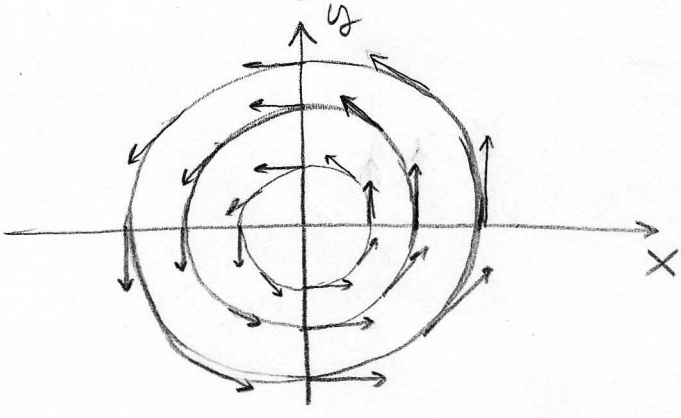
$$x = \frac{dy}{dt} = \frac{d}{dt}\left(A\sin(t+\varphi)\right) = \boxed{A\cos(t+\varphi) = x}$$

We find that $\vec{F} = \langle -y, x \rangle$ has integral curve $\vec{r}(t) = \langle A\cos(t+\varphi), A\sin(t+\varphi) \rangle$.

We can rewrite this,

$$\vec{r}(t) = A\langle \cos(t+\varphi), \sin(t+\varphi) \rangle$$

It's clear $|\langle \cos(t+\varphi), \sin(t+\varphi) \rangle| = 1$ so $|\vec{r}(t)| = A$ for each $t \in \mathbb{R}$. This is a <u>circle of radius A</u> centered at the origin.



Sketch of $\vec{F}$ and how it relates to a few token flow lines.

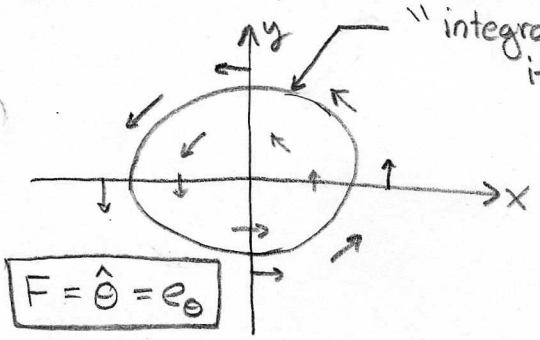
Now perhaps this reminds you of e_θ.

$$\vec{F}(x,y) = \langle -y, x \rangle = \langle -r\sin\theta, r\cos\theta \rangle$$
$$= r\langle -\sin\theta, \cos\theta \rangle$$
$$= re_\theta \quad \text{(see pg. 361)}$$

We see <u>$\vec{F}(r,\theta) = re_\theta$</u>. It is quite natural that we found circles as the flow-lines for this vector field.

(Could find Clockwise sol^{n}s to $\frac{dy}{dx} = \frac{-x}{y}$ just the same way. There is no sense of orientation for the circle here)

E122 Continued, or the flow line of $\vec{F}$ is another perspective see §17.1 #35



"integral curve of the direction field" it is a sol^n of $\frac{dy}{dx} = \frac{-x}{y}$

the little arrows are not derivived, (they're a choice) from the formula $dy/dx = -x/y = \frac{\text{rise}}{\text{run}}$

its $\approx$ $\vec{F}(x,y) = \langle -y, x\rangle$ although perhaps $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$ is better here.

- flow lines of a vector field are paths whose tangents match the vector field

Standard Basis for Vectors in Polar Coordinates

We wish to describe unit vectors that point in the direction of increasing r and θ. Notice first how we could find our familiar friends $\hat{i}$ and $\hat{j}$:

$$\hat{i} = \langle 1, 0\rangle = \nabla x \equiv \hat{x}$$
$$\hat{j} = \langle 0, 1\rangle = \nabla y \equiv \hat{y}$$

observe that the gradients operation reveals just the vector we're looking for. Recall $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$

$$\nabla r = \frac{1}{2\sqrt{x^2+y^2}}\langle 2x, 2y\rangle = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j}$$

$$\nabla\theta = \nabla\left(\tan^{-1}(y/x)\right) = \frac{1}{1+(y/x)^2}\left[\nabla(y/x)\right] = \frac{1}{1+y^2/x^2}\left[\frac{-y}{x^2}\hat{i} + \frac{1}{x}\hat{j}\right]$$

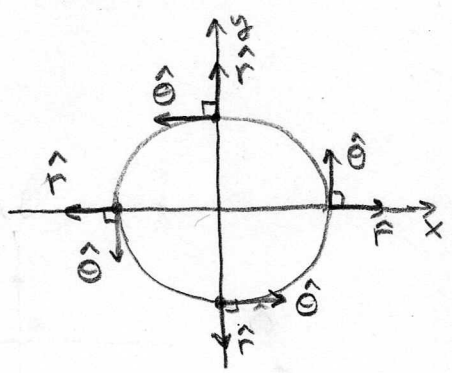
$$\Rightarrow \nabla\theta = \frac{x^2}{x^2+y^2}\left[\frac{-y}{x^2}\hat{i} + \frac{x}{x^2}\hat{j}\right] = \frac{-y}{r^2}\hat{i} + \frac{x}{r^2}\hat{j}$$

We need unit vectors. It may help to use $x = r\cos\theta$, $y = r\sin\theta$,

$$\nabla r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\nabla\theta = \frac{1}{r}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

Clearly $|\nabla r| = \cos^2\theta + \sin^2\theta = 1$, on the other hand $|\nabla\theta| = 1/r$. We find then the Standard Polar basis is

$$\hat{r} \equiv e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{\theta} \equiv e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

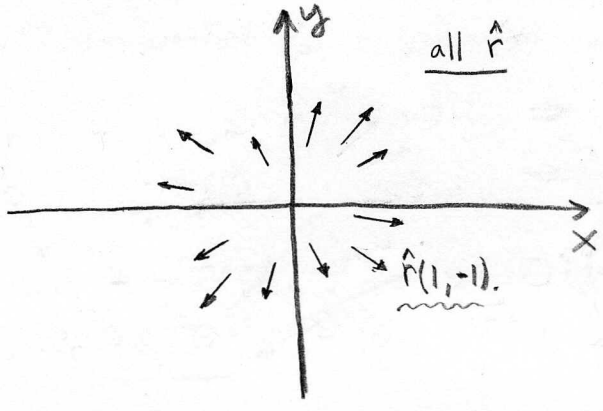


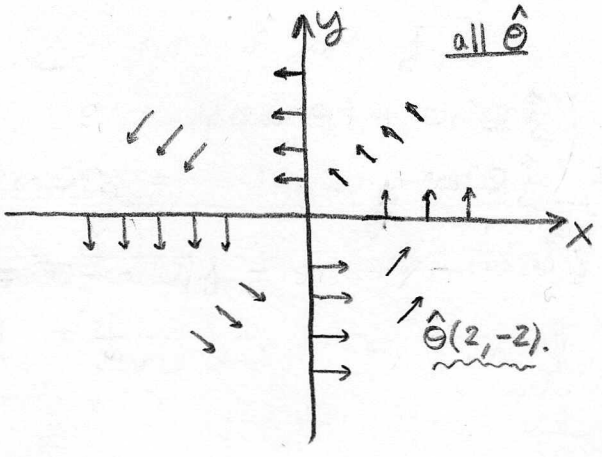
A basis is called orthogonal if the members are perpendicular. Note $\hat{r}\cdot\hat{\theta} = 0$.

e_r and e_θ are vector fields!

The Polar Coordinates Standard Basis : e_r, e_θ

Before I have been quite reckless exchanging points and vectors. This bothered many of you (with good cause, they're different!) It was a special quirk of CARTESIAN COORDINATES that the Standard (CARTESIAN) Basis $\hat{i}, \hat{j}$ is constant over the whole xy-plane. Now the Polar Basis or "frame" is coordinate dependent, we cannot simply move $\hat{\theta}$ or $\hat{r}$ around, their directions are given by θ.





Cylindrical Standard Basis

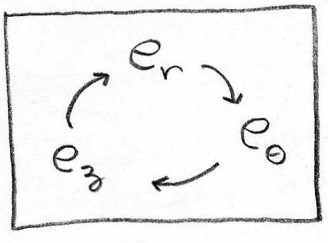
Cylindrical coordinates are right-handed since $e_z = e_r \times e_\theta$

$$e_z \stackrel{?}{=} e_r \times e_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= (\cos^2\theta + \sin^2\theta)\hat{k}$$

$$= \hat{k} = e_z = \hat{z} = \langle 0,0,1 \rangle$$



Cross product works the same on these as $\hat{i}, \hat{j}, \hat{k}$

We can work with vector fields of the form

$$F = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = F_r e_r + F_\theta e_\theta + F_z e_z$$

we would like to be able to use either coordinate system. Obvious question: how do the component functions F_x, F_y, F_z relate to F_r, F_θ, F_z? Its a simple matter of algebra, we tackle it next,

Cylindrical verses Cartesian, converting

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = F_r e_r + F_\theta e_\theta + F_z e_z$$

Clearly the F_z is shared by both, not surprising. Now use (361) to write

$$F_x \hat{i} + F_y \hat{j} = F_r(\cos\theta \hat{i} + \sin\theta \hat{j}) + F_\theta(-\sin\theta \hat{i} + \cos\theta \hat{j})$$
$$= (F_r \cos\theta - F_\theta \sin\theta)\hat{i} + (F_r \sin\theta + F_\theta \cos\theta)\hat{j}$$

Therefore we find

$$\boxed{\begin{aligned} F_x &= F_r \cos\theta - F_\theta \sin\theta \\ F_y &= F_r \sin\theta + F_\theta \cos\theta \end{aligned}} \quad (*)$$

In matrix notation

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta\, F_x + \sin\theta\, F_y \\ -\sin\theta\, F_x + \cos\theta\, F_y \end{pmatrix}$$

Thus the inverse relations are,

$$\boxed{\begin{aligned} F_r &= \cos\theta\, F_x + \sin\theta\, F_y \\ F_\theta &= -\sin\theta\, F_x + \cos\theta\, F_y \end{aligned}} \quad (**)$$

You could derive these relations geometrically if you prefer. Or we could use $F_r = \vec{F} \cdot e_r$ and $F_\theta = \vec{F} \cdot e_\theta$ and that'd be quicker.

E123 Suppose $\vec{F} = \langle 1, 2, 3 \rangle$ find $\vec{F}$ in cylindrical frame.

$$F_r = \cos\theta\, F_x + \sin\theta\, F_y = \cos\theta + 2\sin\theta$$
$$F_\theta = -\sin\theta\, F_x + \cos\theta\, F_y = -\sin\theta + 2\cos\theta$$
$$F_z = F_z = 3$$

Thus $\boxed{\vec{F} = (\cos\theta + 2\sin\theta)e_r + (2\cos\theta - \sin\theta)e_\theta + 3e_z}$

the components are coordinate dependent even though the original $\vec{F}$ was constant, somehow these conspire to be constant. Consider $\theta = 0$ and $\theta = \pi/2$ for example,

$$\vec{F}|_{\theta=0} = (e_r + 2e_\theta + 3e_z)|_{\theta=0} = \hat{i} + 2\hat{j} + 3\hat{k} = \langle 1, 2, 3 \rangle.$$
$$\vec{F}|_{\theta=\pi/2} = (2e_r - e_\theta + 3\hat{k}|_{\theta=\pi/2} = 2\hat{j} - (-\hat{i}) + 3\hat{k} = \langle 1, 2, 3 \rangle.$$

SPHERICAL STANDARD BASIS

Spherical coordinates are $\rho = \sqrt{x^2+y^2+z^2}$ and θ, φ where θ is the same polar angle as we discussed in the cylindrical case.

Thus $e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \frac{-y}{r}\hat{i} + \frac{x}{r}\hat{j}$ where $r = \sqrt{x^2+y^2}$

and e_ρ is easily calculated,

$$e_\rho = \frac{\nabla\rho}{|\nabla\rho|} = \frac{x}{\rho}\hat{i} + \frac{y}{\rho}\hat{j} + \frac{z}{\rho}\hat{k} = \cos\theta\sin\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\varphi\,\hat{k}$$

finally construct e_φ as $-e_\rho \times e_\theta$:

oops should have done $e_\varphi = e_\theta \times e_\rho$ but I already finished this

$$-e_\varphi = e_\rho \times e_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= (-\cos\varphi\cos\theta)\hat{i} - (\cos\varphi\sin\theta)\hat{j} + (\cos^2\theta\sin\varphi + \sin^2\theta\sin\varphi)\hat{k}$$

$$= \underline{-\cos\varphi\cos\theta\,\hat{i} - \cos\varphi\sin\theta\,\hat{j} + \sin\varphi\,\hat{k} = -e_\varphi}$$

I would hope that $e_\varphi = \frac{\nabla\varphi}{|\nabla\varphi|}$. (You can check it).

To summarize:

$$e_\rho = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \cos\theta\sin\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\varphi\,\hat{k}$$

$$e_\theta = \frac{-y\hat{i}+x\hat{j}}{\sqrt{x^2+y^2}} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad (*)$$

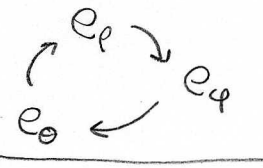
$$e_\varphi = \frac{xz\hat{i}+yz\hat{j}-(x^2+y^2)\hat{k}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} = \cos\varphi\cos\theta\,\hat{i} + \cos\varphi\sin\theta\,\hat{j} - \sin\varphi\,\hat{k}$$

Then if $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta$ we have

$$F_\rho = \vec{F}\cdot e_\rho = \cos\theta\sin\varphi\,F_x + \sin\theta\sin\varphi\,F_y + \cos\varphi\,F_z$$

$$F_\theta = \vec{F}\cdot e_\theta = -\sin\theta\,F_x + \cos\theta\,F_y \qquad (**)$$

$$F_\varphi = \vec{F}\cdot e_\varphi = \cos\varphi\cos\theta\,F_x + \cos\varphi\sin\theta\,F_y - \sin\varphi\,F_z$$



$e_\rho \times e_\varphi = e_\theta$
$e_\varphi \times e_\theta = e_\rho$
$e_\theta \times e_\rho = e_\varphi$

Spherical are Right-Handed also.

SPHERICAL VECTOR BASIS CONTINUED

I'd like to solve (364 **) for F_x, F_y, F_z. I'm going to use (364 *) and the fact $F_x = \vec{F}\cdot\hat{i}$, $F_y = \vec{F}\cdot\hat{j}$ and $F_z = \vec{F}\cdot\hat{k}$ to do this

$$\begin{aligned} F_x &= (F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta)\cdot\hat{i} \\ &= F_\rho(e_\rho\cdot\hat{i}) + F_\varphi(e_\varphi\cdot\hat{i}) + F_\theta(e_\theta\cdot\hat{i}) \\ &= \cos\theta\sin\varphi\, F_\rho + \cos\theta\cos\varphi\, F_\varphi - \sin\theta\, F_\theta \end{aligned}$$

using (*) on (364)

$$\begin{aligned} F_y &= F_\rho(e_\rho\cdot\hat{j}) + F_\varphi(e_\varphi\cdot\hat{j}) + F_\theta(e_\theta\cdot\hat{j}) \\ &= \sin\theta\sin\varphi\, F_\rho + \sin\theta\cos\varphi\, F_\varphi + \cos\theta\, F_\theta \end{aligned}$$

using (*) on (364)

$$\begin{aligned} F_z &= F_\rho(e_\rho\cdot\hat{k}) + F_\varphi(e_\varphi\cdot\hat{k}) + F_\theta(e_\theta\cdot\hat{k}) \\ &= \cos\varphi\, F_\rho - \sin\varphi\, F_\varphi \end{aligned}$$

Thus we find

$$\boxed{\begin{aligned} F_x &= \cos\theta\sin\varphi\, F_\rho + \cos\theta\cos\varphi\, F_\varphi - \sin\theta\, F_\theta \\ F_y &= \sin\theta\sin\varphi\, F_\rho + \sin\theta\cos\varphi\, F_\varphi + \cos\theta\, F_\theta \\ F_z &= \cos\varphi\, F_\rho - \sin\varphi\, F_\varphi \end{aligned}} \quad (*)$$

Remark: the reason I've gone to the trouble of calculating these relations is that it is a good warm-up to calculating the explicit formulas for gradient in other coordinates. Also it will help us find useful NON-CARTESIAN formulas for the "Curl" and "Divergence". Lets address the question of gradients for now.