

CALCULUS OF VECTOR-VALUED FUNCTIONS OF A REAL VARIABLE

(263)

We switch gears a little and return to study parametrized curves like $\vec{r}(t)$. Notice $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$ it is the assignment

$$t \longmapsto \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

the key here is that the domain is \mathbb{R} (or generally some subset) so we can do all the usual calculus; limits, derivatives & integrals. We simply do them componentwise. Notice $\vec{r}(t)$ is actually a vector of three functions from $\mathbb{R} \rightarrow \mathbb{R}$, these functions $x(t), y(t), z(t)$ are the component functions.

Defn/ Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ then

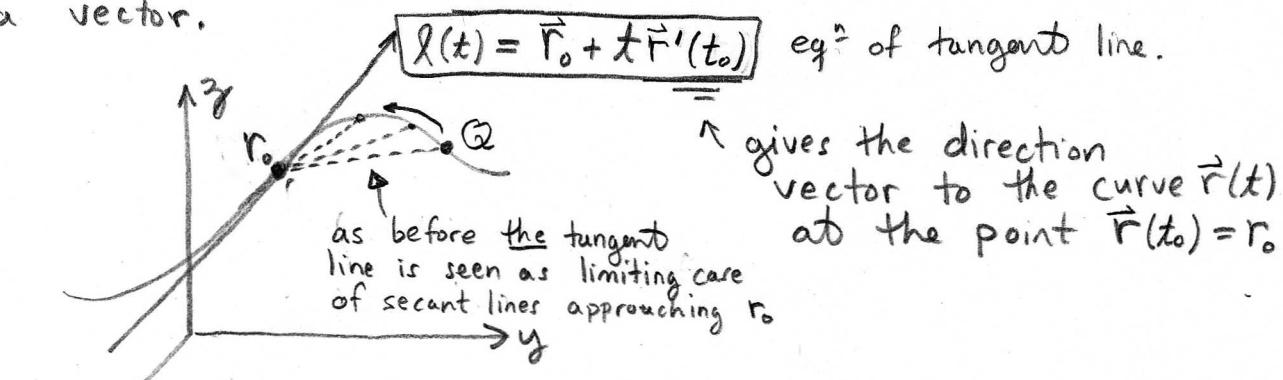
$$\lim_{t \rightarrow a} \vec{r}(t) = \left\langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right\rangle$$

$$\frac{d\vec{r}}{dt} \equiv \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle$$

Likewise for one sided limits and indefinite integrals.

Remark: ok, so I made Thm^m(2) the defn because I can. The geometric interpretation of $\vec{r}'(t)$ as the slope of the tangent line at $\vec{r}(t)$ needs a little generalization, afterall $\vec{r}'(t)$ is a vector.



$$\frac{d\vec{r}}{dt} = \lim_{h \rightarrow 0} \underbrace{\left(\frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right)}$$

direction vector of secant line.

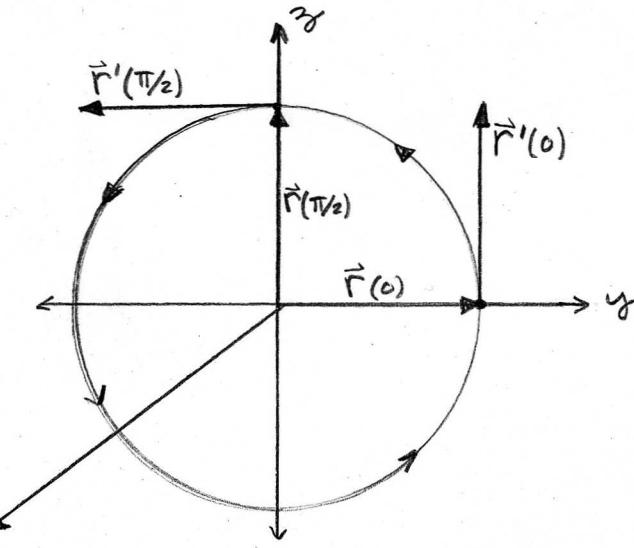
E26 Consider $\vec{r}(t) = \langle 0, \cos t, \sin t \rangle$. Find $\vec{r}'(t)$ and plot $\vec{r}'(0)$ and $\vec{r}'(\pi/2)$. Then calculate $\int \vec{r}(t) dt$.

$$\begin{aligned}\vec{r}'(t) &= \frac{d}{dt}(\langle 0, \cos t, \sin t \rangle) \\ &= \left\langle \frac{d}{dt}(0), \frac{d}{dt}(\cos t), \frac{d}{dt}(\sin t) \right\rangle \\ &= \boxed{\langle 0, -\sin t, \cos t \rangle = \frac{d\vec{r}}{dt}}\end{aligned}$$

$$\begin{aligned}\int \vec{r}(t) dt &= \left\langle \int 0 dt, \int \cos t dt, \int \sin t dt \right\rangle \\ &= \langle C_1, \sin t + C_2, -\cos t + C_3 \rangle \\ &= \boxed{\langle 0, \sin t, -\cos t \rangle + C = \int \vec{r}(t) dt}\end{aligned}$$

Evaluate:

$$\begin{aligned}\vec{r}(0) &= \langle 0, 1, 0 \rangle \\ \vec{r}'(0) &= \langle 0, 0, 1 \rangle \\ \vec{r}(\pi/2) &= \langle 0, 0, 1 \rangle \\ \vec{r}'(\pi/2) &= \langle 0, -1, 0 \rangle\end{aligned}$$

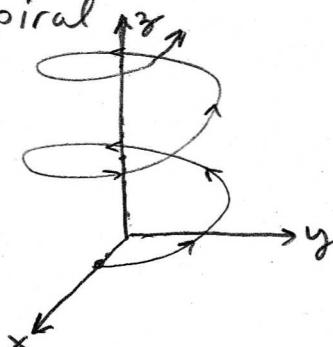


We see that $\vec{r}(t)$ is a circle of radius one in the yz -plane.

Algebraically we can prove this,

$$\begin{aligned}x &= 0 \\ y &= \cos t \\ z &= \sin t \\ \Rightarrow y^2 + z^2 &= \sin^2 t + \cos^2 t = 1\end{aligned}$$

E27 Consider $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. Plot and find $\vec{r}'(t)$. This projects to the circle $\cos^2 t + \sin^2 t = x^2 + y^2 = 1$ in the xy -plane, its a spiral



$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

notice $\frac{dz}{dt} = 1$ so this spiral rises with slope one for all time.

Remark: to give a complete description we should give $\vec{r}(t)$ as well as the allowed values for t . This becomes more important as we go on.

E28 Study $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ where \vec{r}_0, \vec{v} are fixed vectors independent of time. Let $\vec{v} = \langle a, b, c \rangle$,

$$\vec{r}'(t) = \frac{d}{dt}(\vec{r}_0 + t\vec{v}) = \cancel{\frac{d\vec{r}_0}{dt}}_0 + \frac{d}{dt}\langle ta, tb, tc \rangle = \langle a, b, c \rangle = \vec{v}.$$

The tangent vector to this curve is $\vec{r}'(t) = \vec{v}$ for all time. We find that a line in \mathbb{R}^3 has constant direction vectors.

PROPERTIES OF $\frac{d}{dt}$ AND \int ON VECTOR VALUED FUNCTIONS OF \mathbb{R}

Let \vec{A} and \vec{B} be vector valued functions of \mathbb{R} ,

$$(i.) \frac{d}{dt}[\vec{A} + \vec{B}] = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$(ii.) \frac{d}{dt}[c\vec{A}] = c \frac{d\vec{A}}{dt}$$

$$(iii.) \frac{d}{dt}[f\vec{A}] = \frac{df}{dt}\vec{A} + f \frac{d\vec{A}}{dt} : \text{where } f \text{ is a real-valued function of } t.$$

$$(iv.) \frac{d}{dt}[\vec{A} \cdot \vec{B}] = \vec{A}'(t) \cdot \vec{B}(t) + \vec{A}(t) \cdot \vec{B}'(t)$$

$$(v.) \frac{d}{dt}[\vec{A} \times \vec{B}] = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

$$(vi.) \frac{d}{dt}[\vec{A}(f(t))] = \frac{d\vec{A}}{dt}(f(t)) \frac{df}{dt} = \vec{A}'(f(t)) f'(t) : \text{Chain Rule}$$

Proof: these would all be reasonable test questions to prove. I'll do several, they all follow from the ordinary linearity of $\frac{d}{dt}$, product rule or in the case of (vi) the chain rule. Denote

$$\vec{A}(t) = \langle A_1(t), A_2(t), A_3(t) \rangle \text{ and } \vec{B}(t) = \langle B_1(t), B_2(t), B_3(t) \rangle.$$

I begin with (ii).

$$\begin{aligned} \frac{d}{dt}(c\vec{A}) &= \frac{d}{dt}\langle cA_1, cA_2, cA_3 \rangle \\ &= \langle \frac{d}{dt}(cA_1), \frac{d}{dt}(cA_2), \frac{d}{dt}(cA_3) \rangle \\ &= \langle c \frac{dA_1}{dt}, c \frac{dA_2}{dt}, c \frac{dA_3}{dt} \rangle \\ &= c \langle A_1'(t), A_2'(t), A_3'(t) \rangle = c \frac{d\vec{A}}{dt} // \end{aligned}$$

definition of $\frac{d}{dt}$ of a vector valued function of t .

Proof: these are more fun if you know Einstein's repeated index notation. In short in that notation we write $\vec{A} = A_i e_i$, $\vec{B} = B_j e_j$, $\vec{A} \cdot \vec{B} = A_i B_i$ and

of course $\vec{A} \times \vec{B} = \epsilon_{ijk} A_i B_j e_k$ where the ϵ_{ijk} is the standard basis of \mathbb{R}^3 , $e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

and $e_i \times e_j = \epsilon_{ijk} e_k$, here ϵ_{ijk} is the antisymmetric symbol. Having concluded our crash course in index trickery (not req'd topic for ma 242!)

$$\begin{aligned} (\text{iii.}) \quad \frac{d}{dt} [f \vec{A}] &= \frac{d}{dt} [f A_i e_i] \\ &= \frac{d}{dt} [f A_i] e_i \quad \leftarrow \text{this is our def'}. \\ &= \left(\frac{df}{dt} A_i + f \frac{dA_i}{dt} \right) e_i : \text{plain-old product rule applied thrice.} \\ &= \frac{df}{dt} \vec{A} + f \frac{d\vec{A}}{dt}. \end{aligned}$$

$$\begin{aligned} (\text{iv.}) \quad \frac{d}{dt} [\vec{A} \cdot \vec{B}] &= \frac{d}{dt} [A_i B_i] \\ &= \frac{dA_i}{dt} B_i + A_i \frac{dB_i}{dt} : \text{again the product rule} \\ &\quad \text{three times once for each value of } i, \\ &= \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}. \end{aligned}$$

$$\begin{aligned} (\text{v.}) \quad \frac{d}{dt} [\vec{A} \times \vec{B}] &= \frac{d}{dt} [\epsilon_{ijk} A_i B_j e_k] \\ &= \left(\frac{d}{dt} \epsilon_{ijk} A_i B_j \right) e_k \\ &= \left(\epsilon_{ijk} \frac{d}{dt} [A_i B_j] \right) e_k \quad \leftarrow \boxed{\epsilon_{ijk} \text{ is constant with respect to } t.} \\ &= \epsilon_{ijk} \left(\frac{dA_i}{dt} B_j + A_i \frac{dB_j}{dt} \right) e_k \\ &= \epsilon_{ijn} \frac{dA_i}{dt} B_j e_n + \epsilon_{ijk} A_i \frac{dB_j}{dt} e_n = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} // \end{aligned}$$

Proof continued: You can prove these by brute-force, that is just write out all the terms. I like the Einstein notation because it allows us to focus on the crucial point which is that all these properties are simply stolen from calculus I due to the linearity of the vector construction, dot-product and cross product.

$$\begin{aligned}
 \text{(vi)} \quad \frac{d}{dt} (\vec{A}(f(t))) &= \frac{d}{dt} [A_j(f(t)) e_j] \\
 &= \left(\frac{d}{dt} [A_j(f(t))] \right) e_j \\
 &= \left(\frac{d A_j}{dt} \Big|_{f(t)} \frac{df}{dt} \right) e_j \\
 &= \left(\frac{df}{dt} \right) \cdot (A'_j(f(t)) e_j) \\
 &= \frac{df}{dt} \vec{A}'(f(t)) \\
 &= \vec{A}'(f(t)) f'(t) = \frac{d\vec{A}}{dt} \Big|_{f(t)} \frac{df}{dt} //
 \end{aligned}$$

- Many unnecessary steps here. but do you know all these notations?

E29 Suppose $\vec{F}(t) = \langle t^3, t^2, t \rangle$ and $g(t) = \underline{\cosh(t)}$.

$$\vec{F}'(t) = \langle 3t^2, 2t, 1 \rangle$$

$$\begin{aligned}
 \cosh(t) &\equiv \frac{1}{2}(e^t + e^{-t}) \\
 \sinh(t) &\equiv \frac{1}{2}(e^t - e^{-t})
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt} [\vec{F}(g(t))] &= \vec{F}'(g(t)) \cdot g'(t) \\
 &= \langle 3\cosh^2(t), 2\cosh(t), 1 \rangle \cdot \sinh(t) \\
 &= \langle 3\sinh(t)\cosh^2(t), 2\sinh(t)\cosh(t), \sinh(t) \rangle
 \end{aligned}$$

Of course you get the same answer if you first compose then differentiate.

$$\begin{aligned}
 \vec{F}(g(t)) &= \langle \cosh^3 t, \cosh^2 t, \cosh t \rangle \\
 \Rightarrow \frac{d}{dt} (\vec{F}(g(t))) &= \langle 3\cosh^2(t)\sinh t, 2\cosh t \sinh t, \sinh t \rangle
 \end{aligned}$$

However, a sufficiently diabolical test maker can easily make the 2nd approach inaccessible.

TANGENT LINES TO CURVE IN \mathbb{R}^3

(268)

Given a curve parametrized by $\vec{r}(t)$ then the tangent line to the curve at $\vec{r}(t_0)$ is parametrized by the line $\vec{l}(t)$

$$\boxed{\vec{l}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)}$$

E30 Consider the curve $\vec{r}(t) = \langle \cos t, \sin t, \frac{1}{100} \sin(100t) \rangle$, $0 \leq t \leq 2\pi$. Plot the curve and the tangent line at $\vec{r}(\pi/4)$. Also find the eqn of the tangent line.

$$\vec{r}'(t) = \langle -\sin t, \cos t, \frac{100}{100} \cos(100t) \rangle$$

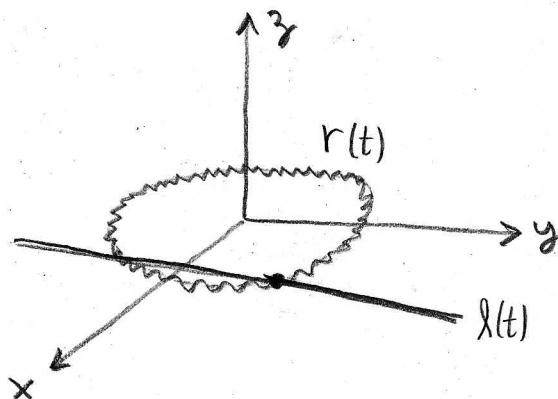
$$\vec{r}(\pi/4) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$$

$$\vec{r}'(\pi/4) = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \cos(25\pi) \rangle = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1 \rangle$$

Thus the eqn of the tangent line to $r(\pi/4)$ is

$$\boxed{\vec{l}(t) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle + t \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1 \rangle}$$

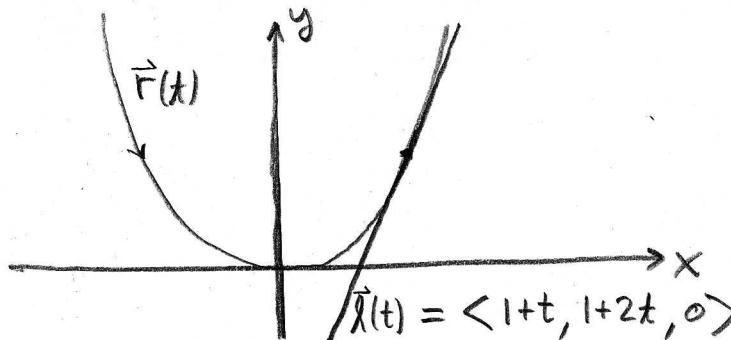
Notice $\frac{1}{100} \leq z \leq \frac{1}{100}$ so $\approx z=0$ and the curve is simply $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Roughly,



Remark: conceptually the "t" in $\vec{r}(t)$ and $\vec{l}(t)$ is distinct. Probably it'd be better to use a different parameter for \vec{l} .

E31 find tangent line to $\vec{r}(t) = \langle t, t^2, 0 \rangle$ at $t=1$.

Note $\vec{r}'(t) = \langle 1, 2t, 0 \rangle$ thus $\boxed{\vec{l}(t) = \langle 1, 1, 0 \rangle + t \langle 1, 2, 0 \rangle}$



the curve and tangent line both lie in $z=0$ so this is easier to picture.