

THE CHAIN RULE FOR FUNCTIONS OF SEVERAL VARIABLES

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To begin we will treat the simple cases where our intermediate variables are independent, later we'll deal with some subtler cases which arise in common applications. First the basics. Assume that

f is differentiable in each of the following.

① If $w = f(x)$ and $x = x(t)$ then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt}$$

② If $w = f(x, y)$ and x, y are functions of t then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

③ If $w = f(x, y, z)$ and x, y, z are functions of t then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

④ If $w = f(x_1, x_2, \dots, x_n)$ and x_1, x_2, \dots, x_n are functions of t then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$$

We know ① to be true from calculus I. Let's give a "proof" of ② since the proofs of ③ & ④ are the same with a bit more writing. We have yet to define differentiability but we can essentially take it to mean that f is approximated by

$$L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

so for (x, y) "close" to (a, b) we have $f(x, y) \approx L(x, y)$. We'll discuss the linearization $L(x, y)$ more later on. For now, consider if $w = f(x, y)$ and let $(x(t_0), y(t_0)) = (a, b)$.

$$\begin{aligned} \frac{dw}{dt} \Big|_{t_0} &= \frac{d}{dt} \left[f(x(t), y(t)) \right] \Big|_{t_0} \\ &= \frac{d}{dt} \left[f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \right] \Big|_{t_0} \\ &= f_x(a, b) \frac{dx}{dt} \Big|_{t_0} + f_y(a, b) \frac{dy}{dt} \Big|_{t_0} \\ \therefore \frac{dw}{dt} &= \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt} \end{aligned}$$

A good technical proof needs E's and S's. Ask me if you're interested.

Examples of $f(x, y)$ or $f(x, y, z)$ where the intermediate variables x, y, z are themselves functions of an independent variable t .

E55 Let $w = xy$ and suppose $x = e^t$ and $y = \sin t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = ye^t + x \cos t = e^t \sin t + e^t \cos t = \boxed{\frac{dw}{dt}}$$

E56 Let $z = xy$ and again suppose $x = e^t$ & $y = \sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = e^t \sin t + e^t \cos t = \boxed{\frac{dz}{dt}}$$

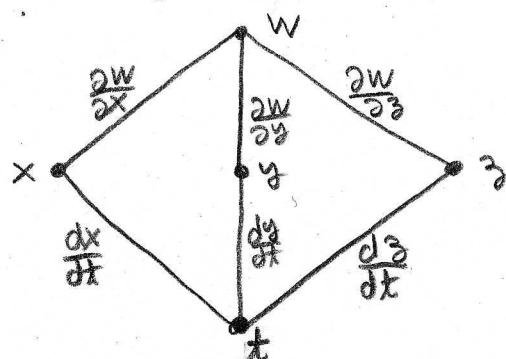
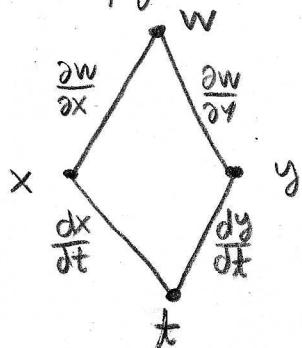
- Sometimes z is taken as the dependent variable. Other times z is playing the role of an intermediate variable.

E57 Let $w = xyz$ and suppose $x = t$, $y = t^2$, $z = t^3$

$$\begin{aligned}\frac{dw}{dt} &= \frac{d}{dt} [w(x(t), y(t), z(t))] \\ &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= yz + xz(2t) + xy(3t^2) \\ &= t^5 + 2t^5 + 3t^5 \\ &= \boxed{6t^5 = \frac{dw}{dt}} \quad (\text{which is good since } w = t^6 \text{ so } \frac{dw}{dt} = 6t^5)\end{aligned}$$

Some people find the following "Tree Diagrams" a help in figuring out the correct chain rule. You may use them if they help

You multiply down then add together the different branches.



dependent

intermediate

independent

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

CHAIN RULE WITH SEVERAL INDEPENDENT VARIABLES (s, t)

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Again assume that f is differentiable, also suppose the intermediate variables x, y, z are functions of s, t (or u, v or u_1, u_2, \dots, u_n)

⑤ If $w = f(x)$ and $x = x(s, t)$ then:

$$\frac{\partial w}{\partial s} = \frac{df}{dx} \frac{\partial x}{\partial s} \quad \& \quad \frac{\partial w}{\partial t} = \frac{df}{dx} \frac{\partial x}{\partial t}$$

⑥ If $w = f(x, y)$ and x, y are functions of $s \& t$ then:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \& \quad \frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

⑦ If $w = f(x, y, z)$ and x, y, z are functions of s, t then:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s} \quad \& \quad \frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$$

⑧ If $w = f(x_1, x_2, \dots, x_n)$ and x_1, x_2, \dots, x_n are functions of s, t then:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial s} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial s} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t}$$

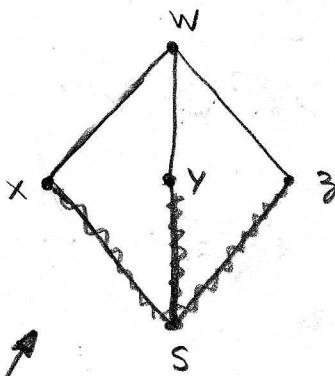
⑨ If $w = f(x_1, x_2, \dots, x_n)$ and x_1, \dots, x_n are facts. of u_1, u_2, \dots, u_k then:

$$\frac{\partial w}{\partial u_1} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial u_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial u_1} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial u_1}$$

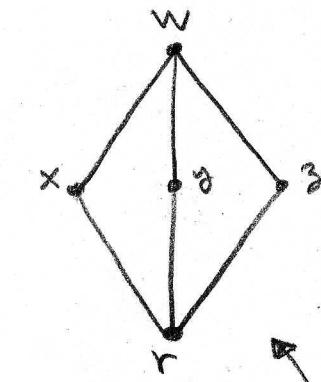
$$\vdots$$

$$\frac{\partial w}{\partial u_k} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial u_k} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial u_k} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial u_k}$$

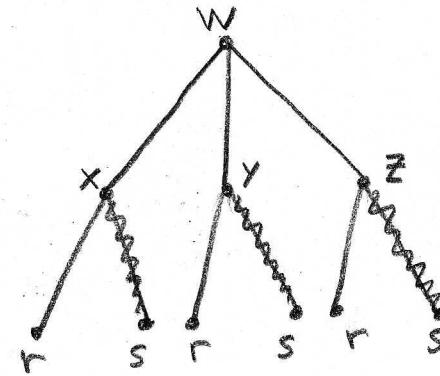
With Tree Diagrams: (I can't do color here so I'll use squiggles)



$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}$$



$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$$



Remark: The proof of $\textcircled{5} \rightarrow \textcircled{9}$ follows from the earlier results

$\textcircled{1} \rightarrow \textcircled{4}$. You simply fix either s or t etc... and so the $\frac{\partial}{\partial t}$'s become $\frac{\partial}{\partial t}$'s. Remember the " ∂ " notation just reminds us that there are possibly several variables which ride along unchanged. You can write for $f = f(x)$ that $\frac{df}{dx} = \frac{\partial f}{\partial x}$ if you wish. However if $f = f(x, y)$ then we must be careful to distinguish the concepts.

E58 Suppose $w = e^x \sin(y)$ and $y = st^2$, $x = \ln(s-t)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} : \text{identified that } w \text{ is function of } x \text{ & } y.$$

$$= e^x \sin(y) \frac{1}{s-t} + e^x \cos(y) t^2 : \text{note } e^x = e^{\ln(s-t)} = s-t.$$

$$= \boxed{\sin(st^2) + (s-t)t^2 \cos(st^2) = \frac{\partial w}{\partial s}}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

$$= e^x \sin(y) \left(\frac{-1}{s-t} \right) + e^x \cos(y) (2st) : \text{again } e^x = s-t,$$

$$= \boxed{-\sin(st^2) + 2st(s-t) \cos(st^2) = \frac{\partial w}{\partial t}}$$

E59 Suppose that $z = f(x, y)$ has continuous $f_x, f_y, f_{xy}, f_{yx}, f_{xx}$ etc--- and $x = r^2 + s^2$ and $y = 2rs$ then note,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y}$$

We wish to compute $\frac{\partial^2 z}{\partial r^2}$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} \left[2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y} \right]$$

$$= 2 \frac{\partial z}{\partial x} + 2 \frac{\partial}{\partial r} \left[\frac{\partial z}{\partial x} \right] + 2s \frac{\partial}{\partial r} \left[\frac{\partial z}{\partial y} \right]$$

$$= 2 \frac{\partial z}{\partial x} + 2 \left[\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial r} \right] + 2s \left[\frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} \right]$$

$$= 2 \frac{\partial z}{\partial x} + 2 \left[\frac{\partial^2 z}{\partial x^2} \cdot 2r + \frac{\partial^2 z}{\partial y \partial x} \cdot 2s \right] + 2s \left[\frac{\partial^2 z}{\partial x \partial y} \cdot 2r + \frac{\partial^2 z}{\partial y^2} \cdot 2s \right]$$

$$= 2f_x + 4rf_{xx} + 4sf_{xy} + 4sr f_{yx} + 4s^2 f_{yy}$$

$$= \boxed{2f_x + 4rf_{xx} + 4s^2 f_{yy} + 4s(1+r)f_{xy}} \text{ using Clairaut's Thm.}$$

(This is the best we can do w/o an explicit formula for $f(x, y)$.)