

## IMPLICIT DIFFERENTIATION

(300)

Upto now we have considered  $x, y$  to be independent (or  $x, y, z$ ) and then we supposed that  $x, y$  had parametrizations in terms of one or two (or more) independent variables  $s, t$ . What if we take a different view, what if we assume that  $x, y, z$  are related implicitly through some relation. To begin we consider the problem :  $F(x, y) = 0$  find  $\frac{dy}{dx}$ . Recall in calc. I we use implicit differentiation to do this for example,

**E60**  $\sin(x)\cos(y) + y^2 = x^3$  suppose  $y = y(x)$  find  $\frac{dy}{dx}$

$$\cos(x)\cos(y) - \sin(x)\sin(y) \frac{dy}{dx} + 2y \frac{dy}{dx} = 3x^2$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - \cos(x)\cos(y)}{2y - \sin(x)\sin(y)}$$

We may arrive at this result through another approach, perhaps easier.

**① SET-UP:** Suppose  $F(x, y) = 0$  implicitly defines  $y = f(x)$  such that  $F(x, f(x)) = 0$ . Now differentiate w.r.t.  $x$ , (here "t" =  $x$ )

$$\frac{dF}{dx} = \frac{d}{dx}(0) = 0 = \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \boxed{-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}} \quad \text{"eq 6" of Stewart, (pg. 941)}$$

**E61** Let's revisit **E60** to begin define  $F(x, y) = x^3 - \sin(x)\cos(y) - y^2 = 0$ ,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(3x^2 - \cos(x)\cos(y))}{\sin(x)\sin(y) - 2y} = \frac{3x^2 - \cos(x)\cos(y)}{2y - \sin(x)\sin(y)}.$$

(this can shortcut a lot of calculation, not so much for my example).

② SET-UP: Suppose  $F(x, y, z) = 0$  implicitly defines  $z = f(x, y)$ , so that  $F(x, y, f(x, y)) = 0$ . Then

$$\frac{dF}{dx} = 0 = \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \quad \therefore \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Likewise we can derive that

$$\frac{dF}{dy} = 0 = \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} \quad \therefore \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

**E62** Consider  $x^2 + y^2 + z^2 = 1$  find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ . We suppose that  $z = z(x, y)$ , define  $F(x, y, z) = x^2 + y^2 + z^2 - 1$ . Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z} = \boxed{-\frac{x}{z}} = \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y}{2z} = \boxed{-\frac{y}{z}} = \frac{\partial z}{\partial y}$$

Remark: the implicit function Th" states that given certain conditions the eq<sup>n</sup>  $F(x, y) = 0$  or  $F(x, y, z) = 0$  gives  $y = y(x)$  or  $z = z(x, y)$ . That is one of the variables is given implicitly as a function of the remaining variables. The one "dependent" variable we took to be  $y$  in **E61** or  $z$  in **E62**, but why not the other way around? Why isn't  $x$  the dependent variable? The answer, it can be. This concept is not just a mathematical digression, there are common engineering/physics applications for which a given set of variables are w/o a preference for choosing "dependent" variable. Probably the most familiar is the eq"

$$PV = nRT$$

here  $P, V, T$  are all variables, which one is the dependent? I'll spend a few pages exposing the danger and introducing a notation to fix the potential pitfall.  
(We are indebted to Thomas' Calc. § 11.9)

## PARTIAL DERIVATIVES OF CONSTRAINED VARIABLES

(302)

This does not seem to appear explicitly in Stewart. I borrow the presentation of Thomas' Calc. 10<sup>th</sup> Ed. §11.9. The topic here has the potential to be confusing but with proper notation the trouble can be avoided. This notation is absent from Stewart see problems 82, 83, 85 of §15.3 and #58 of §15.5; he takes an indirect approach by mentioning the independent/dependent variables in the set-up for the problem. I think it's better to be a bit more explicit about this issue. Let's begin with what can go wrong if we are not careful. (Also see p. 157 #21 → 24 of Colley.)

### CAUTIONARY EXAMPLE:

Find  $\frac{\partial w}{\partial x}$  given that  $w = x^2 + y^2 + z^2$  and  $z = x^2 + y^2$

We have two eq's and four unknowns. Clearly since we are asked to find  $\frac{\partial w}{\partial x}$  this suggests  $w$  is a dependent variable whereas  $x$  is independent. This leaves  $y$  and  $z$ , one of these must be dep. the other independent.

That is we either write

$$w = w(x, y) \quad \text{or} \quad w(x, z)$$

Suppose  $w = w(x, y, z(y))$ :  $x, y$  are independent

$$\begin{aligned} w &= x^2 + y^2 + z^2 : \text{need to leave just } x \text{ & } y \\ &= x^2 + y^2 + (x^2 + y^2)^2 : \text{substitute } z = x^2 + y^2 \\ &= x^2 + y^2 + x^4 + 2x^2y^2 + y^4 \end{aligned}$$

$$\therefore \boxed{\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2}$$

Suppose  $w = w(x, y(z), z)$ :  $x, z$  are independent

$$\begin{aligned} w &= x^2 + y^2 + z^2 : \text{need to eliminate } y \\ &= x^2 + z - x^2 + z^2 : \text{solve } z = x^2 + y^2 \text{ for } y^2 = z - x^2. \\ &= z + z^2 \end{aligned}$$

$$\therefore \boxed{\frac{\partial w}{\partial x} = 0} \quad ! ? (?) \text{ etc...}$$

this is why the independent / dependent variables must be made explicit.

Notational Cure:

$$\left( \frac{\partial w}{\partial x} \right)_y = 2x + 4x^3 + 4xy^2$$

$$\left( \frac{\partial w}{\partial x} \right)_z = 0$$

Bonus Points: expose the geometric meaning of  $\left(\frac{\partial w}{\partial x}\right)_y \neq \left(\frac{\partial w}{\partial x}\right)_z$  found in the "Cautionary Example". It's not that complicated we just don't have time for it.

ADVICE: How to find  $\frac{\partial w}{\partial x}$  when  $w = F(x, y, z)$  has inputs constrained by another equation

- ① Decide which variables are dependent or independent.
- ② Eliminate the other dependent variables in  $w$ .
- ③ Differentiate as usual.

Let me further illustrate with some real-world examples

**E63** Suppose that  $PV = nRT$  where  $n, R$  are constants and  $P$  = pressure,  $V$  = volume,  $T$  = temperature. This is the Ideal Gas Law. Problem 82 of §15.3 asks us to show that  $\left(\frac{\partial P}{\partial V}\right)_{T,P} \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1$ . Let's clarify that, we need to show  $\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1$ .

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{\partial}{\partial V} \left[ \frac{nRT}{V} \right] \Big|_{T \text{ fixed}} = nRT \left( -\frac{1}{V^2} \right) : P = P(V, T)$$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{\partial}{\partial T} \left[ \frac{nRT}{P} \right] \Big|_{P \text{ fixed}} = \frac{nR}{P} : V = V(T, P)$$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{\partial}{\partial P} \left[ \frac{PV}{nR} \right] \Big|_{V \text{ fixed}} = \frac{V}{nR}$$

Now assemble these and remember  $PV = nRT$ ,

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = \left(-\frac{nRT}{V^2}\right) \left(\frac{nR}{P}\right) \left(\frac{V}{nR}\right) = -\frac{nRT}{PV} = \frac{-nRT}{nRT} = -1.$$

Remark: this hawk problem is really confusing if you don't settle the question of what's a function of what! Trust me.

**E64** Lets look at #85 of §15.3

Given  $K = \frac{1}{2}mv^2$  show that  $\frac{\partial K}{\partial m} \frac{\partial^2 K}{\partial v^2} = K$ . To begin lets make this problem statement more precise, show

$$\left(\frac{\partial K}{\partial m}\right)_v \left(\frac{\partial^2 K}{\partial v^2}\right)_m = K$$

very well, lets begin

$$\left(\frac{\partial K}{\partial m}\right)_v = \frac{\partial}{\partial m} \left[ \frac{1}{2}mv^2 \right] \Big|_{v-\text{fixed}} = \frac{1}{2}v^2$$

$$\left(\frac{\partial K}{\partial v}\right)_m = \frac{\partial}{\partial v} \left[ \frac{1}{2}mv^2 \right] \Big|_{m-\text{fixed}} = mv \quad (\text{momentum!})$$

$$\left(\frac{\partial^2 K}{\partial v^2}\right)_m = \frac{\partial}{\partial v} [mv] \Big|_{m-\text{fixed}} = m.$$

Therefore we find that  $K = \frac{1}{2}mv^2 = \boxed{\left(\frac{\partial K}{\partial m}\right)_v \left(\frac{\partial^2 K}{\partial v^2}\right)_m = K}$ .

Remark: the interesting examples are found in Thermodynamics. I'll put off those until we discuss total differentials.

**E65** Again suppose  $PV = NRT$ , but this time suppose that only  $R$  is constant so  $P, V, N, T$  are variables.

$$\left(\frac{\partial P}{\partial T}\right)_{V,N} = \frac{\partial}{\partial T} \left[ \frac{NRT}{V} \right] \Big|_{V,N \text{ fixed}} = \frac{NR}{V}$$

$$\left(\frac{\partial T}{\partial V}\right)_{P,N} = \frac{\partial}{\partial V} \left[ \frac{PV}{NR} \right] \Big|_{P,N \text{ fixed}} = \frac{P}{NR}$$

$$\left(\frac{\partial V}{\partial N}\right)_{P,T} = \frac{\partial}{\partial N} \left[ \frac{NRT}{P} \right] \Big|_{P,T \text{ fixed}} = \frac{RT}{P}$$

$$\left(\frac{\partial N}{\partial P}\right)_{T,V} = \frac{\partial}{\partial P} \left[ \frac{PV}{RT} \right] \Big|_{T,V \text{ fixed}} = \frac{V}{RT}$$

$$\left(\frac{\partial P}{\partial T}\right)_{V,N} \left(\frac{\partial T}{\partial V}\right)_{P,N} \left(\frac{\partial V}{\partial N}\right)_{P,T} \left(\frac{\partial N}{\partial P}\right)_{T,V} = \frac{NR}{V} \frac{P}{NR} \frac{RT}{P} \frac{V}{RT} = 1.$$

Curious. I have no idea what this means. Enlighten me if you know.

**E66** Suppose  $U = f(P, V, T) =$  internal energy of a gas that obeys the Ideal Gas Law  $PV = nRT$  ( $n, R$  constants).

$$\left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial}{\partial P} [f(P, V, T)] \Big|_{V-\text{fixed}} = \frac{\partial f}{\partial P} \frac{\partial P}{\partial P} + \frac{\partial f}{\partial T} \frac{\partial T}{\partial P} = \boxed{\frac{\partial f}{\partial P} + \frac{\partial f}{\partial T} \frac{V}{nR}}$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial}{\partial T} [f(P, V, T)] \Big|_{V-\text{fixed}} = \frac{\partial f}{\partial P} \frac{\partial P}{\partial T} + \frac{\partial f}{\partial T} \frac{\partial T}{\partial T} = \boxed{\frac{\partial f}{\partial P} \left(\frac{nR}{V}\right) + \frac{\partial f}{\partial T}}$$

I've used  $T = PV/nR$  and  $P = nRT/V$  to calculate  $\partial T/\partial P \neq \partial P/\partial T$ . And it's better to write  $\frac{\partial U}{\partial P} + \frac{\partial U}{\partial T} \frac{V}{nR} = \left(\frac{\partial U}{\partial P}\right)_V$  and  $\frac{\partial U}{\partial P} \frac{nR}{V} + \frac{\partial U}{\partial T} = \left(\frac{\partial U}{\partial T}\right)_V$ .

**E67** Suppose that  $x^2 + y^2 = r^2$  and  $x = r\cos\theta$ ,  $y = r\sin\theta$ ,

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \frac{\partial}{\partial r} [r\cos\theta] \Big|_{\theta-\text{fixed}} = \cos\theta$$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{\partial}{\partial x} [r] \Big|_{y-\text{fixed}} = \frac{\partial}{\partial x} [\sqrt{x^2+y^2}] \Big|_{y-\text{fixed}} = \frac{x}{\sqrt{x^2+y^2}}$$

Just trying to elucidate the notation.

**E68** Show if  $f(x, y, z) = 0$  then  $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$ . We begin by exploiting  $f(x, y, z) = 0$  to give a few differential relations,

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} \quad \therefore \left(\frac{\partial z}{\partial x}\right)_y = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \left(\frac{\partial y}{\partial x} = 0\right)$$

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} \quad \therefore \left(\frac{\partial x}{\partial y}\right)_z = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} \quad \left(\frac{\partial z}{\partial y} = 0\right)$$

$$\frac{\partial f}{\partial z} = 0 = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial f}{\partial z} \quad \therefore \left(\frac{\partial y}{\partial z}\right)_x = \frac{-\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} \quad \left(\frac{\partial x}{\partial z} = 0\right)$$

Thus,

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{f_x}{f_y}\right) \left(-\frac{f_z}{f_x}\right) \left(-\frac{f_x}{f_z}\right) = -1.$$

Remark: hmm... this example seems quite close to one of the homework problems.