

# CARTESIAN COORDINATES

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We begin by settling some notations to be used throughout the course,

$$\mathbb{R} \equiv (-\infty, \infty)$$

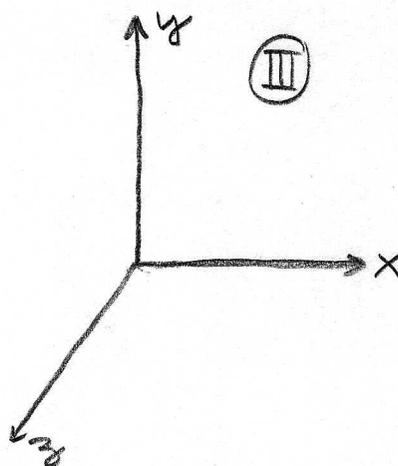
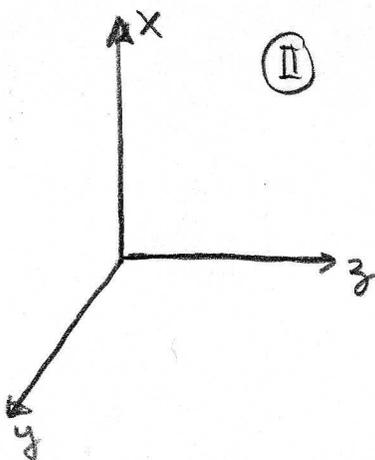
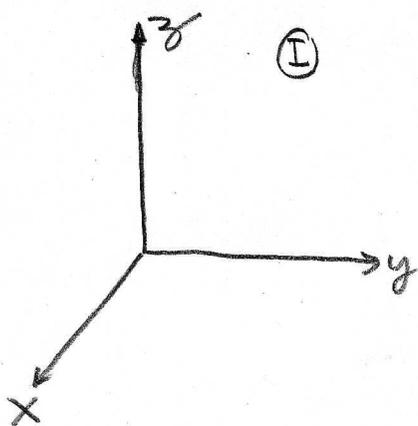
$$\mathbb{R}^2 \equiv \{(x, y) \mid x, y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$$

$$\mathbb{R}^3 \equiv \{(x, y, z) \mid x, y, z \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

“Cartesian Products”

$$\mathbb{R}^n \equiv \{(x_1, x_2, \dots, x_n) \mid n \text{ a positive integer, } x_i \in \mathbb{R} \text{ } i=1, 2, \dots, n\}$$

An element of  $\mathbb{R}^n$  is called an “ $n$ -tuple” or a “point”. In the cases of  $n=1, 2$  or  $3$  we can identify these spaces with our everyday motions where height, width and depth are essential concepts. You are already familiar with graphs and coordinates in  $n=1$  and  $2$ , but perhaps  $n=3$  is new,



I will almost always use  $\textcircled{\text{I}}$ , but  $\textcircled{\text{II}}$  &  $\textcircled{\text{III}}$  are just a different view of the same. These are all “right handed coordinate systems”. We will define that carefully via cross products a little later. There are also left handed coordinate systems, but we will not use them.

# Coordinates and Projections

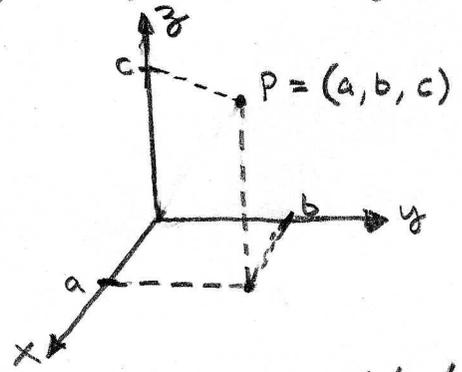
Given a point  $P = (a, b, c) \in \mathbb{R}^3$  we define the the

$P_1 \equiv P_x = a$  the x-coordinate of P

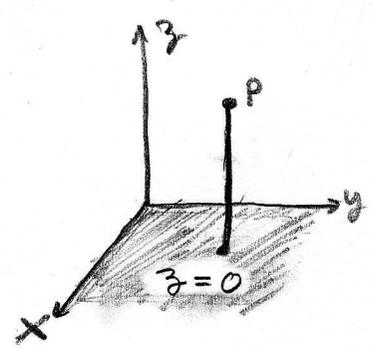
$P_2 \equiv P_y = b$  the y-coordinate of P

$P_3 \equiv P_z = c$  the z-coordinate of P

Graphically, assuming  $a, b, c > 0$  (P is in the 1<sup>st</sup> octant)

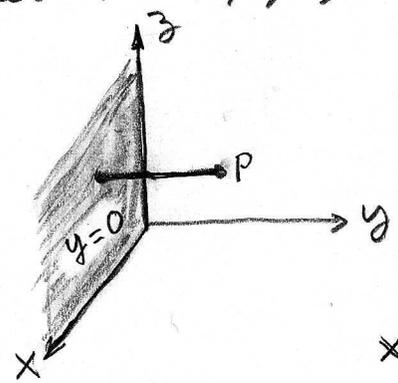


Often it will be useful to employ projections onto the coordinate planes. Let  $P = (a, b, c)$  as before,



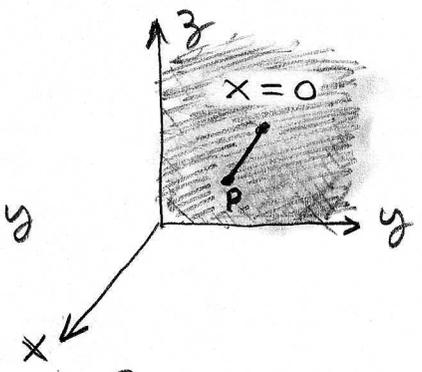
Projection onto the xy-plane

$$\pi_{xy}(P) = (a, b, 0)$$



Projection onto the xz-plane

$$\pi_{xz}(P) = (a, 0, c)$$



Projection onto the yz-plane

$$\pi_{yz}(P) = (0, b, c)$$

Remark: In  $\mathbb{R}^2$  the eq<sup>n</sup>s  $x=0$  or  $y=0$  would have given us a vertical or horizontal line, but in the context of  $\mathbb{R}^3$  they give planes because for each value of  $z$  we get a line. If you paste a bunch of lines together they'll make a plane. (provided they're lined up correctly, oh sorry.)

# Distance between Points & Line Segments

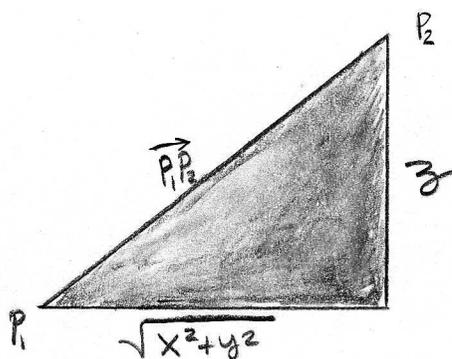
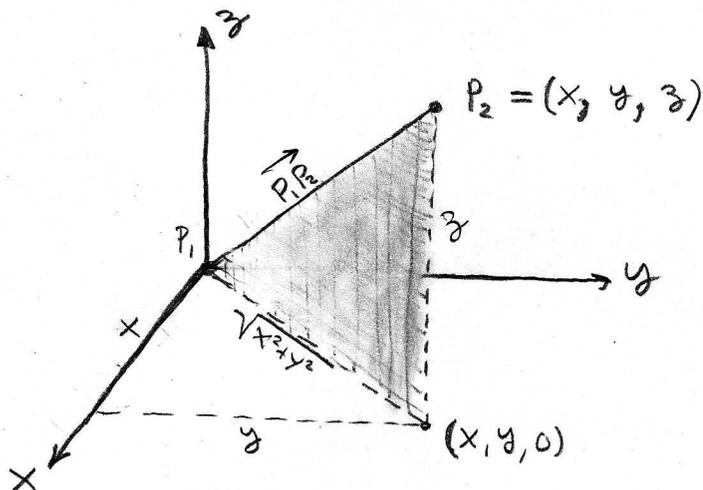
Let  $P_1 = (x_1, y_1, z_1)$  and  $P_2 = (x_2, y_2, z_2)$  then the directed line segment from  $P_1$  to  $P_2$  is a vector

$$\vec{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \equiv P_2 - P_1$$

The distance between  $P_1$  &  $P_2$  is the length of the line segment connecting them. We can show,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = |\vec{P_1 P_2}|$$

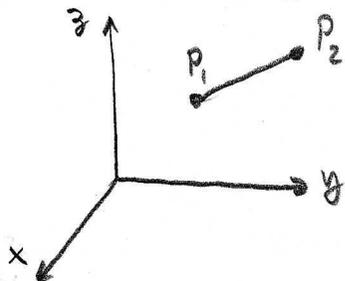
Lets put  $P_1$  at the origin so  $P_1 = (0, 0, 0)$  and  $P_2 = (x, y, z)$



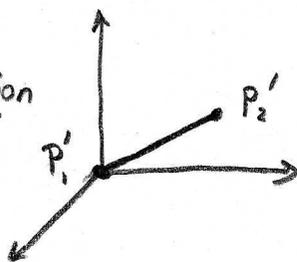
We apply the pythagorean Th<sup>m</sup> to the right triangle in (xy)-plane. Then again apply the pythagorean Th<sup>m</sup> to the shaded  $\Delta$ ,

$$|\vec{P_1 P_2}| = \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

Remark: it is sufficient to prove this for  $P_1 = (0, 0, 0)$  since if  $P_1 \neq (0, 0, 0)$  then we could translate both  $P_1$  and  $P_2$  by  $-P_1$  so that  $P_1' = (0, 0, 0)$  and  $P_2' = P_2 - P_1$ . If we shift both points simultaneously then the same directed line segment connects them (although it's based at the origin instead of  $P_1$ .)



translation  
by  $-P_1$



$T(P) = P - P_1$   
translation  
by  $-P_1$   
operation.

## Length of line segment in $n$ -dimensions

Let  $P_1 = (x_i)$  and  $P_2 = (y_i)$  for  $i=1, 2, \dots, n$  then

$$|\vec{P_1 P_2}| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2}$$

Examples: choose  $P_1 = 0$

$$n=1 \quad |y| = \sqrt{y^2}$$

$$n=2 \quad |(y_1, y_2)| = \sqrt{y_1^2 + y_2^2}$$

$$n=3 \quad |(y_1, y_2, y_3)| = \sqrt{y_1^2 + y_2^2 + y_3^2}$$

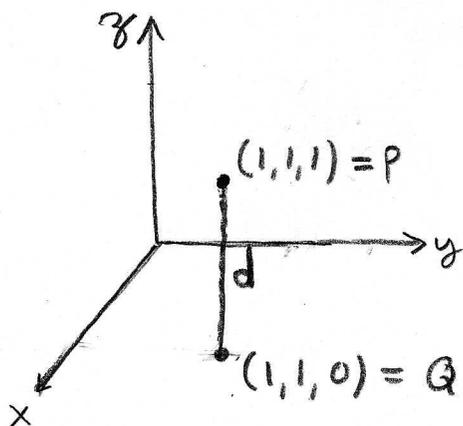
Remark: this is the Euclidean type of distance. There are other ideas.  $\mathbb{R}^n$  equipped with this idea of distance is called Euclidean Space. It is the geometry natural to Classical Newtonian Mechanics.

**E1**: THE SPHERE OF RADIUS  $R$  based at  $(a, b, c) = P$  is defined to be the collection of all  $\vec{r} = (x, y, z) \in \mathbb{R}^3$  such that  $|\vec{r} - P| = R > 0$ . That is

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = R$$

$$\Rightarrow \boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2} \quad \text{Sphere of radius } R \text{ at } (a, b, c).$$

**E2** Find distance from point  $(1, 1, 1)$  to  $xy$ -plane. This is by common agreement the distance to the closest point on the  $xy$ -plane to  $(1, 1, 1)$ . We can see this must be the distance  $d$  in the picture.



$$|P - Q| = |(1, 1, 1) - (1, 1, 0)| = |(0, 0, 1)| = \sqrt{1^2} = \boxed{1}$$

Question: how to do this if we had an arbitrary tilted plane? Its not quite so easy, is it? We will learn tools to help with this.