

THE CURL OF A VECTOR FIELD:

We now find the next use of our friend " ∇ " (del or nabla). We are shipping ahead to §17.5 since it is our custom to differentiate before we integrate, and because it allows me to better explain certain topics in §17.2, 17.3, 17.4 and so on...

Defⁿ Let $\vec{F} = \langle P, Q, R \rangle$ where each component function is differentiable, and $\vec{F} = \langle F_1, F_2, F_3 \rangle$ as well,

$$\text{curl } (\vec{F}) \equiv \nabla \times \vec{F}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

$$= \langle \partial_z F_3 - \partial_y F_2, \partial_x F_1 - \partial_z F_3, \partial_y F_2 - \partial_x F_1 \rangle$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

E127 Let $\vec{F} = \langle x, y, z \rangle$ find $\nabla \times \vec{F}$.

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y), \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z), \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right\rangle = 0.$$

Hmm... notice $\vec{F} = \nabla \left[\frac{1}{2}(x^2 + y^2 + z^2) \right]$. So \vec{F} is conservative.

E128 Let $\vec{F} = \langle -y, x, 0 \rangle$ we saw this in **E122** on 361

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \hat{i}(\partial_y(0) - \partial_z(x)) - \hat{j}(\partial_x(0) - \partial_z(y)) + \hat{k}(\partial_x(x) + \partial_y(0))$$

$$\Rightarrow \boxed{\nabla \times \vec{F} = 2\hat{k}}$$

Using the $\vec{F} = \langle P, Q, R \rangle$ notation we have $P = -y$ & $Q = x$ you might note $\frac{\partial P}{\partial y} = -1$ whereas $\frac{\partial Q}{\partial x} = 1$ thus

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. We'll learn a little later this means $\nexists U$ such that $\vec{F} = \nabla U$. That is \vec{F} is not "conservative"

Remark: We can deduce a few facts about the "curl" operation. It measures if the vector field twists or curls about some point. In E107 we found $\nabla \times \vec{F} = 0$ for a vector field which could be written $\vec{F} = \rho e_p$, so it's purely "radial" it diverges straight away from the origin. On the other hand in E108 we found $\nabla \times \vec{F} = 2\hat{k}$ for the vector field $\vec{F} = \langle -y, x, 0 \rangle$ which circles the origin in the xy-plane. We could say the curl measures the circulation of a vector field. Finally, we noticed that if $\vec{F} = \nabla U$ then $\nabla \times \vec{F} = \nabla \times (\nabla U) = 0$. Intuitively this is appealing since $\vec{A} \times \vec{A} = 0$, however we know that results for an ordinary vector, not for a vector of operators which is what $\nabla = \hat{i}\partial_x + \hat{j}\partial_y + \hat{k}\partial_z$ is. It is true though, we'll prove it.

Thⁿ/ If g has continuous 2nd order partials then $\nabla \times \nabla g = 0$

Proof: follows from direct calculation + Clairaut's Thⁿ,

$$\begin{aligned}\nabla \times \nabla g &= \nabla \times [\langle \partial_x g, \partial_y g, \partial_z g \rangle] \\ &= \langle \partial_y(\partial_z g) - \partial_z(\partial_y g), \partial_z(\partial_x g) - \partial_x(\partial_z g), \partial_x(\partial_y g) - \partial_y(\partial_x g) \rangle \\ &= \left\langle \frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 g}{\partial z \partial y}, \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 g}{\partial x \partial z}, \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y \partial x} \right\rangle \\ &= \langle 0, 0, 0 \rangle \quad \text{(applying } \partial_i \partial_j g = \partial_j \partial_i g \text{ for continuous fnct. } g.\text{)}\end{aligned}$$

Corollary: If \vec{F} is conservative then $\nabla \times \vec{F} = 0$

this means we can verify that \vec{F} fails to be conservative if $\nabla \times \vec{F} \neq 0$. It is not however guaranteed that if $\nabla \times \vec{F} = 0$ then \vec{F} is conservative.

$$\vec{F} \text{ conservative} \Rightarrow \nabla \times \vec{F} = 0 \text{ (always)}$$

$$\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \underbrace{\nabla U}_{\text{it turns out the topology matters.}} \text{ (sometimes)}$$

Remark: \vec{F} is said to be irrotational iff $\nabla \times \vec{F} = 0$

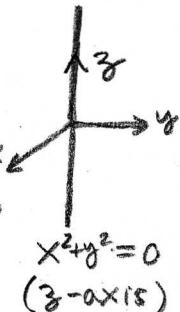
Thⁿ If \vec{F} is a vector field defined on all of \mathbb{R}^3 whose components functions have continuous partial derivatives and $\text{curl}(\vec{F}) = 0$ then \vec{F} is a conservative vector field

a stronger version says this Thⁿ holds for simply-connected domains of \vec{F} . Simply-connected is a topological concept that means the space has no holes.

E129 Let $\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$. Lets check if \vec{F} is conservative. Later we'll show $\nabla \times \vec{F} = 0$. Thus we should be able to find f such that $\vec{F} = \nabla f$ meaning,

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} = \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2} = \frac{\partial}{\partial x} \left[\tan^{-1}(y/x) \right]$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{\partial}{\partial y} \left[\tan^{-1}(y/x) \right]$$



Thus $f = \tan^{-1}(y/x)$. Here's the catch, $\text{dom}(\vec{F}) = \{(x,y,z) \mid x^2+y^2 \neq 0\}$ whereas $\text{dom}(f) = \{(x,y,z) \mid x \neq 0\}$ thus \vec{F} is not conservative despite the fact $\nabla \times \vec{F} = 0$. This example is a subtle one and ultimately it leads to many deep advances in modern quantum field theory (it's the magnetic monopole)

E130 Let $\vec{F} = \langle 2x+y, 3\cos(yz)+x, y\cos(yz) \rangle$. Is \vec{F} conservative, if so find its potential function f such that $\vec{F} = \nabla f$. We can test if \vec{F} is conservative by examining $\nabla \times \vec{F}$, the $\text{dom}(\vec{F}) = \mathbb{R}^3$ so we'll not have to worry about the exception to the Thⁿ, in this case: $\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F}$ conservative

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y & 3\cos(yz)+x & y\cos(yz) \end{vmatrix}$$

$$= \hat{i} (\cos(yz) - 3y\sin(yz)) + \hat{j} (-\cos(yz) + 3y\sin(yz)) + \hat{k} (0 - 0)$$

$$= \underline{0} = \nabla \times \vec{F} \Rightarrow \text{since } \text{dom}(\vec{F}) = \mathbb{R}^3 \text{ we find } \vec{F} \text{ conservative.}$$

E130 Continued: We found that \vec{F} is conservative with the help of the $\nabla \times \vec{F}$ test. In principle this is not necessary, we alternatively could simply assume that $\exists f$ such that $\vec{F} = \nabla f$. If \vec{F} is not conservative we'll not be successful in finding the potential function.

$$\nabla f = \langle 2x + y, 3\cos(yz) + x, y\cos(yz) \rangle$$

Yields three partial differential eq's, (PDEs)

$$\frac{\partial f}{\partial x} = 2x + y, \quad \frac{\partial f}{\partial y} = 3\cos(yz) + x, \quad \frac{\partial f}{\partial z} = y\cos(yz)$$

generally PDE's aren't easy to solve, but these are easy. I'll illustrate a procedure you should know (this comes up in other math courses like Differential Eq's and applications)

$$f = \int \frac{\partial f}{\partial x} dx = \int (2x + y) dx = x^2 + xy + C_1(y, z)$$

I've used the notation " ∂x " to emphasize that we hold $y \neq z$ constant in the integration, the constant with respect to x can be a function of $y \neq z$ as I emphasize with the notation $C_1(y, z)$. Now we'll use the remaining two PDEs to pin down the explicit form of $C_1(y, z)$.

$$\frac{\partial f}{\partial y} = 3\cos(yz) + x = \frac{\partial}{\partial y} [x^2 + xy + C_1(y, z)] = x + \frac{\partial C_1}{\partial y}$$

$$\Rightarrow 3\cos(yz) = \frac{\partial C_1}{\partial y}$$

$$\Rightarrow C_1 = \int \frac{\partial C_1}{\partial y} dy = \int 3\cos(yz) dy = \frac{3\sin(yz)}{z} + C_2(z).$$

Notice that $C_1 = C_1(y, z)$ so I knew that $C_2 = C_2(z)$, it can only depend on z otherwise C_1 might acquire an x -dependence.

$$\frac{\partial f}{\partial z} = y\cos(yz) = \frac{\partial}{\partial z} [x^2 + xy + \sin(yz) + C_2(z)]$$

$$\Rightarrow y\cos(yz) = y\cos(yz) + \frac{\partial C_2}{\partial z} \Rightarrow \frac{dC_2}{dz} = 0 \therefore \underline{C_2 = \text{constant}}$$

Here $\frac{\partial C_2}{\partial z} = \frac{dC_2}{dz}$ since C_2 only depends on z . In total,

$$f = x^2 + xy + \sin(yz) + C_2 \quad \text{is the potential function for } F$$

Remark: this seems longer than it is. I've put extra comments here to try to clarify the method. Also see §13.5 #13 & 16 for more examples.