

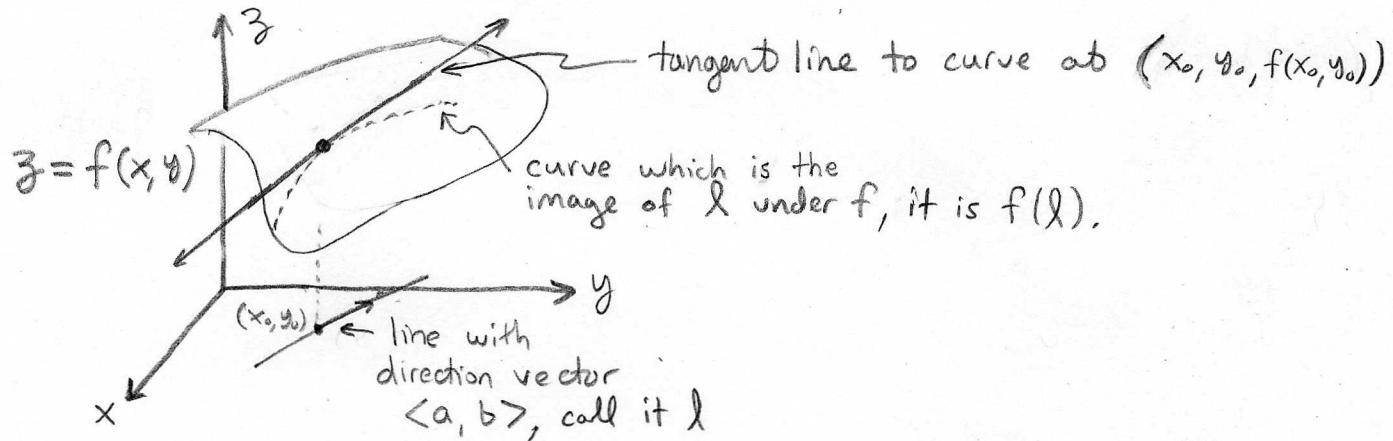
# THE GRADIENT VECTOR AND DIRECTIONAL DERIVATIVES

(314)

The directional derivative of  $f$  at  $(x_0, y_0)$  in the direction  $\hat{u} = \langle a, b \rangle$  is the rate of change in  $f$  in that direction from  $(x_0, y_0)$ ,

$$D_{\hat{u}} f(x_0, y_0) \equiv \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Here the notation  $D_u f(x_0, y_0)$  is usually employed, but take note that  $|u| = 1$  is required. The geometry of this is



In the special cases  $\langle a, b \rangle = \langle 1, 0 \rangle$  or  $\langle 0, 1 \rangle$  we obtain plain-old partial derivatives, it's the same idea of tangency to  $z = f(x, y)$  just tilted. That is,

$$D_{\langle 1, 0 \rangle} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \equiv f_x(x_0, y_0).$$

$$D_{\langle 0, 1 \rangle} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \equiv f_y(x_0, y_0).$$

Proposition:  $D_{\langle a, b \rangle} f(x_0, y_0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b$

Proof: See p. 948, it's an easy consequence of the chain rule.

Def<sup>n</sup>/ The gradient of  $f$  is denoted  $\text{grad}(f)$  or  $\nabla f$  and for  $f: \mathbb{X} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  it is defined by

$$\text{grad}(f) = \nabla f = \langle f_x, f_y \rangle$$

Or at a point  $(\nabla f)(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$ .

Observation:  $D_{\langle a, b \rangle} f(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \langle a, b \rangle$

this is a nice way  
to remember it.

EXAMPLES OF  $\nabla f$  and directional derivatives

**E75** Let  $f(x, y) = \ln(xy)$ . Find the rate of change in  $f$  at the point  $(1, 2)$  in the  $\langle 1, -1 \rangle$  direction. To begin we calculate the gradient,

$$\nabla f = \left\langle \frac{\partial}{\partial x}[\ln(xy)], \frac{\partial}{\partial y}[\ln(xy)] \right\rangle = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$$

notice  $\langle 1, -1 \rangle$  has length  $\sqrt{1^2 + (-1)^2} = \sqrt{2}$  so the unit vector we need here is  $u = \frac{1}{\sqrt{2}}\langle 1, -1 \rangle$ . So calculate,

$$\begin{aligned} D_u f(1, 2) &= \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{1}{\sqrt{2}} - \frac{1}{2} \left( \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{2} \right) = \frac{1}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}} \end{aligned}$$

in geometric terms this says the slope of the curve on  $z = \ln(xy)$  above  $\vec{r}(t) = \langle 1, 2 \rangle + t\langle 1, -1 \rangle$  at  $(1, 2, \ln(2))$  has slope  $\sqrt{2}/4$ . This is how quickly  $f$  changes in the  $\langle 1, -1 \rangle$  direction at  $(1, 2)$ .

• WHAT  $\langle a, b \rangle$  give min/max rates of change at  $(1, 2)$ ?

Recall that  $\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$  and  $-1 \leq \cos \theta \leq 1$  so we obtain max. when  $\theta = 0$  and min. when  $\theta = \pi$ .

$$\Delta f_{\max} = |\nabla f(1, 2)| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \boxed{\frac{\sqrt{5}}{2}} \text{ max rate.}$$

$$\Delta f_{\min} = -|\nabla f(1, 2)| = -\frac{\sqrt{5}}{2}.$$

We find that  $\langle 1, 1/2 \rangle$  gives max rate of change of  $f$  at  $(1, 2)$  while  $\langle -1, -1/2 \rangle$  gives minimum rate of change of  $f$  at  $(1, 2)$ .

**E76** Let  $f(x, y) = x \sin(y) + y \cos(x)$  find rate of change in the  $\langle 1, 1 \rangle$  direction at  $(\pi, \pi)$ . Note  $u = \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$ ,

$$D_u f(\pi, \pi) = \left\langle \sin(y) - y \sin(x), x \cos(y) + \cos(x) \right\rangle \Big|_{(\pi, \pi)} \cdot \left( \frac{1}{\sqrt{2}} \langle 1, 1 \rangle \right)$$

$$= \frac{1}{\sqrt{2}} [(\sin \pi - \pi \sin \pi) 1 + (\pi \cos \pi + \cos \pi) 1]$$

$$= \boxed{-\frac{1}{\sqrt{2}}(\pi + 1)}$$

SUMMARY:  $D_{\vec{u}} f(x_0, y_0)$  gives the rate of change of  $f$  at  $(x_0, y_0)$  in the  $\vec{u}$ -direction. If  $\vec{u} = \langle a, b \rangle$  then

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = f_x(x_0, y_0) a + f_y(x_0, y_0) b$$

The max/min rates of change at  $(x_0, y_0)$  occur in the  $\pm \nabla f(x_0, y_0)$  directions, with values  $\pm |\nabla f(x_0, y_0)|$ .

### DIRECTIONAL DERIVATIVES OF FUNCTIONS OF THREE OR MORE VARIABLES

The gradient vector for  $f: \mathbb{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

Of course we usually encounter  $n=2$  or  $3$ ,  $n=2$  we've discussed and used many times,  $n=3$  is basically the same

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \text{ where } f = f(x, y, z)$$

again assumed to have length one.

The rate of change of  $f: \mathbb{X} \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$  at  $P$  in the  $\langle a, b, c \rangle$  direction is simply a  $a, b, c$  weighted sum of the change of  $f$  at  $P$  in the  $x, y, z$ -directions, we call this the DIRECTIONAL DERIVATIVE of  $f$  in the  $\langle a, b, c \rangle$  direction at the point  $P = (P_1, P_2, P_3)$

$$D_{\langle a, b, c \rangle} f(P) \equiv (\nabla f)(P_1, P_2, P_3) \cdot \langle a, b, c \rangle$$

again it should be emphasized  $|\langle a, b, c \rangle| = 1$  is assumed here.

E77 Let  $f(x, y, z) = x^2 + y^2 + z^2$ . Find the maximum rate of change at the point  $(1, 1, 1)$ . To begin find  $\nabla f$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla f(1, 1, 1) = \langle 2, 2, 2 \rangle$$

Then  $D_{\vec{u}} f(1, 1, 1) = \langle 2, 2, 2 \rangle \cdot \vec{u}$ , this is maximized when  $\vec{u} \parallel \nabla f$ . We need to find unit vector, so divide by the length of  $\langle 2, 2, 2 \rangle$  to obtain

$$\vec{u} = \frac{1}{\sqrt{12}} \langle 2, 2, 2 \rangle = \boxed{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \vec{u}}$$

- we see in higher dimensions the directional derivative works the same, I will not try to graph this situation as it is in 4-dimensional space,