

MAXIMIZING & MINIMIZING FUNCTIONS OF SEVERAL VARIABLES

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We generalize the terminology of calc. I to fit our purposes here, the meaning of local min/max and absolute min/max have the obvious meanings.

Defⁿ / $f : \text{dom}(f) \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is on some disk centered at (a, b) . Likewise $f(a, b)$ is a local minimum of f if $f(x, y) \geq f(a, b)$ when (x, y) is on some disk centered at (a, b) . If we have that $f(x, y) \leq f(a, b) \quad \forall (x, y) \in S \subseteq \text{dom}(f)$ then we say that $f(a, b)$ is the maximum of f on S . Likewise if $f(x, y) \geq f(a, b) \quad \forall (x, y) \in S \subseteq \text{dom}(f)$ then $f(a, b)$ is the minimum of f on S . When $S = \text{dom}(f)$ we call those a global maximum or minimum.

let me list the theoretical tools then we'll do a few examples.

Thⁿ / If f has a local max/min at (a, b) and the first-order partial derivatives exist there then $(\nabla f)(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = 0$.

Defⁿ / say (a, b) is a critical point if either $(\nabla f)(a, b) = 0$ or one or both of the partial derivatives do not exist,

as the theorem indicates if we have a local max/min at which the partials exist then that point must be a critical point. We cannot reverse this though, just because (a, b) is a critical point that does not guarantee that $f(a, b)$ is a max/min. Just as in the $y = f(x)$ case we'll want to check all the critical points to see if they're extremal.

Thⁿ / Suppose the 2nd partial derivatives of f are continuous on a disk centered on (a, b) and $(\nabla f)(a, b) = 0$. Define

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

then we have three cases

- (i.) If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is local min.
- (ii.) If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is local max.
- (iii.) If $D < 0$, then $f(a, b)$ is not a local max/min. We say that $f(a, b)$ is a saddle point of f in this case.

Examples:

E81 Find local max/min and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

$\nabla f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle = 0$ at critical points.

$$\begin{aligned}\nabla f = 0 \rightarrow 4x^3 - 4y &= 0 \rightarrow y = x^3 \\ \nabla f = 0 \rightarrow 4y^3 - 4x &= 0 \rightarrow x = y^3\end{aligned} \rightarrow x = x^9$$

$$\text{Thus } x^9 - x = x(x^8 - 1) = x(x^4 + 1)(x^4 - 1) = \underbrace{x(x^4 + 1)}_1 \underbrace{(x^2 + 1)}_1 \underbrace{(x^2 - 1)}_1 = 0$$

give real roots $x = 0, \pm 1$.

We find critical points $(0, 0), (1, 1), (-1, 1)$. Now apply the 2nd derivative test Thm, but first find D ,

$$D(x, y) = f_{xx}f_{yy} - [f_{xy}]^2 = (12x^2)(12y^2) - [-4]^2 = 144x^2y^2 - 16.$$

We organize our results in a table

Critical Point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	1	0	$-16 < 0$	saddle point
$(1, 1)$	-1	12	$128 > 0$	local min.
$(-1, 1)$	-1	12	$128 > 0$	local min.

E82 find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + 3 = 4$.

Define $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$ the distance from $(1, 0, -2)$ to (x, y, z)
then if on the plane we have $z = 4 - x - 2y$ so

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

$$\text{Now minimize } f(x, y) = d^2 = (x-1)^2 + y^2 + (6-x-2y)^2$$

$$f_x = 2(x-1) - 2(6-x-2y) = 4x - 14 + 4y$$

$$f_y = 2y - 4(6-x-2y) = 10y - 24 + 4x$$

Critical points have $f_x = 0$ and $f_y = 0$ so, (f_x, f_y) exist continuously everywhere

E82 continued We need $f_x = 0$ and $f_y = 0$ for critical point, this amounts to two eq's & two unknowns here,

$$\begin{cases} 4x + 10y = 24 \\ 4x + 4y = 14 \end{cases}$$

$$6y = 10 \therefore y = 10/6 = 5/3$$

$$\text{then } x = 6 - \frac{10y}{4} = 6 - \frac{10}{4} \frac{10}{6} = 6 - \frac{100}{24} = 6 - \frac{25}{6} = \frac{36-25}{6} = \frac{11}{6}$$

the critical point is $(\frac{11}{6}, \frac{5}{3})$. Now find D,

$$D = f_{xx}f_{yy} - [f_{xy}]^2 = (4)(10) - (4)^2 = 24 > 0 \text{ and } f_{xx} = 4 > 0$$

thus we have a local minimum. So the closest point

$$\text{is where } x = \frac{11}{6}, y = \frac{5}{3} \text{ and } z = 4 - \frac{11}{6} - \frac{10}{3} = \frac{96-44-80}{24} = \frac{-28}{24} = -\frac{7}{6}$$

that is $(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6})$ is the closest point on the plane

$z = 4 - x - 2y$ to the point $(1, 0, -2)$. The distance is

$$d = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{7}{6} + 2\right)^2} = \sqrt{\frac{25 + 100 + 25}{36}} = \sqrt{\frac{6(25)}{36}} = \boxed{\frac{5\sqrt{6}}{6}}$$

Remark: You might wonder how do I maximize a function of three or more variables? The answer is not found in Stewart or Thomas for that matter. Colley has the answer, take a look at §4.2 p. 251 the "Second derivative Test for local extrema". We use $D = f_{xx}f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ the test for functions of three variables is based of the "Hessian" of the function,

$$Hf = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} \quad \text{determinant, like the cross-product formula.}$$

take a look in Colley for the details, its not too tricky.

Absolute Maximums and Minimums

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In calc I. we found a procedure for locating the max/min of $f(x)$ on a closed interval $[a, b]$. We now discuss the generalization of that to $f(x, y)$ for closed and bounded subsets of \mathbb{R}^2 . A closed set contains its boundary points and a bounded set fits inside some finite disk in \mathbb{R}^2 .

Thⁿ If $f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function on a closed bounded set D in \mathbb{R}^2 then f attains an absolute maximum value and an absolute minimum value somewhere in D .

Advice: to find extreme values for continuous f on D we

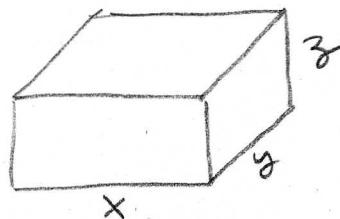
- 1.) find the critical points (where $\nabla f = 0$ or ∇f d.n.e.) and evaluate
 - 2.) find extreme values of f on the boundary of D (which I call ∂D)
 - 3.) The values from 1.) and 2.) compare and choose biggest/smallest.
- (see hwk for example)

E83 Consider a company that accepts only rectangular boxes whose length and girth (the perimeter of a cross section) do not sum over 108". Find the dimensions of an acceptable box of largest volume.

$$V = xyz$$

$$x + \underbrace{2y + 2z}_{\text{length girth}} = 108$$

$$x = 108 - 2y - 2z$$



$$V(y, z) = (108 - 2y - 2z)yz = 108yz - 2y^2z - 2yz^2$$

$$\nabla_y = 108z - 4yz - 2z^2 = 0 \quad \text{looking for critical points.}$$

$$\nabla_z = 108y - 2y^2 - 4yz = 0$$

$$z(108 - 4y - 2z) = 0$$

$$y(108 - 2y - 4z) = 0$$

Now we need these to be simultaneously zero. Notice $z=0$ and $y=0$ gives one solution. Or we could have $108 - 4y - 2z = 0$ and $108 - 2y - 4z = 0$. Or we could have $z=0$ and $108 - 2y - 4z = 0$ or $y=0$ and $108 - 4y - 2z = 0$.

E83 continued) We found $(0,0)$ our first critical point. There are three more,

$$\begin{array}{l} \text{(i)} \quad 108 - 4y - 2z = 0 \Rightarrow \begin{cases} 216 - 8y - 4z = 0 \\ 108 - 2y - 4z = 0 \end{cases} \\ 108 - 2y - 4z = 0 \\ 108 - 6y = 0 \therefore y = \frac{108}{6} = \frac{54}{3} = 18. \end{array}$$

$$z = 54 - 2y = 54 - 36 = 18 = z$$

$\therefore (18, 18)$ another critical pt.

$$\text{(ii.) } z = 0$$

$$108 - 2y - 4z = 0 \Rightarrow 108 - 2y = 0 \Rightarrow y = 54 \therefore (54, 0) \text{ critical pt.}$$

$$\text{(iii.) } y = 0$$

$$108 - 2z - 4y = 0 \Rightarrow 108 - 2z = 0 \Rightarrow z = 54 \therefore (0, 54) \text{ critical point}$$

we have exposed that $(\nabla V)(y, z) = 0$ yields 4 sol's, $(0, 0), (18, 18)$

$(0, 54)$ and $(54, 0)$, these points must give the min/max. values. Calculate

$$V_{yy} = -4z \quad V_{yz} = 108 - 4z$$

$$V_{zz} = -4y$$

$$D = 16yz - (108 - 4z)^2$$

Notice $V(0,0) = 0$, $V(0,54) = 0$ and $V(54,0) = 0$ thus we suspect that $(18, 18)$ gives maximum volume, let's check

$$D(18, 18) = 16(18)^2 - (108 - 72)^2 = 16(18)^2 - (36)^2 = 16(18)^2 - 4(18)^2 = 12(18)^2 > 0$$

and $V_{yy} = -4(18) < 0 \therefore V(18, 18)$ is maximum.

Notice $x = 108 - 2y - 2z = 108 - 36 - 36 = 36$ thus

$$V = xyz = (36)(18)(18) = 11,664 \text{ in}^3 = \boxed{\text{max volume subject to the constraint } x = 108 - 2y - 4z}$$

Remark: we just maximized a function of three variables x, y, z subject to a constraint. Our method was to substitute the constraint then treat it as a 2 variable min/max problem. There is a clever system for the problem in general, the method of Lagrange Multipliers, our next topic.