

Th³/ The gradient is found to be:

$$(1.) \nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta : \text{ for } f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(2.) \nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z : \text{ for } f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$(3.) \nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r \sin \phi} \frac{\partial f}{\partial \phi} e_\phi : \text{ for } f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

• Notice ∇f is a vector field on \mathbb{R}^2 or \mathbb{R}^3 here.

Pf: We begin with case (1.). Recall that $x = r \cos \theta$ and $y = r \sin \theta$ and we found $e_r = \cos \theta \hat{i} + \sin \theta \hat{j}$ & $e_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}$. We need to solve for \hat{i} and \hat{j} in terms of e_r & e_θ , I'll use a standard trick,

$$\hat{i} = a e_r + b e_\theta : \text{ find } a, b \text{ by using dot-product.}$$

$$= (\hat{i} \cdot e_r) e_r + (\hat{i} \cdot e_\theta) e_\theta : \text{ generally } A = (A \cdot \hat{i}) \hat{i} + (A \cdot \hat{j}) \hat{j}$$

$$= \cos \theta e_r - \sin \theta e_\theta = \hat{i}$$

I'm just applying this idea to the polar basis.

$$\hat{j} = (\hat{j} \cdot e_r) e_r + (\hat{j} \cdot e_\theta) e_\theta : \text{ same idea as in } \hat{i}.$$

$$= \sin \theta e_r + \cos \theta e_\theta = \hat{j}$$

Consider them,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \\ &= \frac{\partial f}{\partial x} (\cos \theta e_r - \sin \theta e_\theta) + \frac{\partial f}{\partial y} (\sin \theta e_r + \cos \theta e_\theta) \\ &= \left(\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right) e_r + \left(-\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} \right) e_\theta \\ &= e_r \left[\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right] + e_\theta \frac{1}{r} \left[-r \sin \theta \frac{\partial f}{\partial x} + r \cos \theta \frac{\partial f}{\partial y} \right] \\ &= e_r \frac{\partial f}{\partial r} + e_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} \\ &= e_r \frac{\partial f}{\partial r} + e_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} = \nabla f \end{aligned}$$

see p. 309 we derived this during our discussion of the chain-rule.

In polar coordinates: $\nabla = e_r \frac{\partial}{\partial r} + \frac{1}{r} e_\theta \frac{\partial}{\partial \theta}$.

Proof Continued:

The proof of (2.) follows immediately from (1.) since in that case $\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = e_r \frac{\partial f}{\partial r} + \frac{1}{r} e_\theta \frac{\partial f}{\partial \theta} + e_z \frac{\partial f}{\partial z}$

since $e_z = \hat{k}$. The proof of (3.) is less obvious. To begin let's rewrite $\hat{i}, \hat{j}, \hat{k}$ in the spherical basis e_p, e_ϕ, e_θ ,

$$\begin{aligned}\hat{i} &= (\hat{i} \cdot e_p) e_p + (\hat{i} \cdot e_\phi) e_\phi + (\hat{i} \cdot e_\theta) e_\theta \\ &= \underbrace{\cos \theta \sin \varphi}_{\text{see 364 (*)}} e_p + \underbrace{\cos \theta \cos \varphi}_{\text{to do the}} e_\phi - \underbrace{\sin \theta}_{\text{dot-products}} e_\theta = \hat{i}\end{aligned}\quad \textcircled{I}$$

$$\begin{aligned}\hat{j} &= (\hat{j} \cdot e_p) e_p + (\hat{j} \cdot e_\phi) e_\phi + (\hat{j} \cdot e_\theta) e_\theta \\ &= \underbrace{\sin \theta \sin \varphi}_{\text{see 364 (*)}} e_p + \underbrace{\sin \theta \cos \varphi}_{\text{to do the}} e_\phi + \underbrace{\cos \theta}_{\text{dot-products}} e_\theta = \hat{j}\end{aligned}\quad \textcircled{II}$$

$$\begin{aligned}\hat{k} &= (\hat{k} \cdot e_p) e_p + (\hat{k} \cdot e_\phi) e_\phi + (\hat{k} \cdot e_\theta) e_\theta \\ &= \underbrace{\cos \varphi}_{\text{see 364 (*)}} e_p - \underbrace{\sin \varphi}_{\text{to do the}} e_\phi = \hat{k}\end{aligned}\quad \textcircled{III}$$

I used (*) of 364 to evaluate the dot products above. Consider,

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \\ &= e_p \left(\cos \theta \sin \varphi \frac{\partial f}{\partial x} + \sin \theta \sin \varphi \frac{\partial f}{\partial y} + \cos \varphi \frac{\partial f}{\partial z} \right) \quad : \text{using } \textcircled{I}, \textcircled{II} \text{ & } \textcircled{III} \\ &\quad + e_\phi \left(\cos \theta \cos \varphi \frac{\partial f}{\partial x} + \sin \theta \cos \varphi \frac{\partial f}{\partial y} - \sin \varphi \frac{\partial f}{\partial z} \right) \quad \text{gathering like terms.} \\ &\quad + e_\theta \left(-\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} \right) \\ &= e_p \left(\frac{\partial x}{\partial p} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial p} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial p} \frac{\partial f}{\partial z} \right) \\ &\quad + \frac{1}{p} e_\phi \left(p \cos \theta \cos \varphi \frac{\partial f}{\partial x} + p \sin \theta \cos \varphi \frac{\partial f}{\partial y} - p \sin \varphi \frac{\partial f}{\partial z} \right) \\ &\quad + \frac{1}{p \sin \varphi} e_\theta \left(-p \sin \theta \sin \varphi \frac{\partial f}{\partial x} + p \cos \theta \sin \varphi \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial f}{\partial z} \right) \\ &= e_p \frac{\partial f}{\partial p} + \frac{1}{p} e_\phi \left(\frac{\partial x}{\partial \phi} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial f}{\partial z} \right) + \frac{e_\theta}{p \sin \varphi} \left(\frac{\partial x}{\partial \theta} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial f}{\partial z} \right) \\ &= \boxed{e_p \frac{\partial f}{\partial p} + \frac{1}{p} e_\phi \frac{\partial f}{\partial \phi} + \frac{1}{p \sin \varphi} e_\theta \frac{\partial f}{\partial \theta} = \nabla f}\end{aligned}$$

Observe in Sphericals

$$\nabla = e_p \frac{\partial}{\partial p} + \frac{e_\phi}{p} \frac{\partial}{\partial \phi} + \frac{e_\theta}{p \sin \varphi} \frac{\partial}{\partial \theta}. \quad (\star)$$

E124 Consider the function $U = -\frac{GmM}{r}$. Let's find the gradient of U where G is the gravitational constant, m is the mass of a planet and M is the mass of the sun which we place at $\rho = 0$ so the distance between M & m is $\rho = \sqrt{x^2 + y^2 + z^2}$. Let's use our Th^m,

$$\nabla U = -GmM \left[e_r \frac{\partial}{\partial r} \left(\frac{1}{r} \right) + \frac{1}{r} e_\theta \frac{\partial}{\partial \theta} \left(\frac{1}{r} \right)^0 + \frac{1}{r \sin \theta} e_\phi \frac{\partial}{\partial \phi} \left(\frac{1}{r} \right)^0 \right]$$

$$\nabla U = \frac{GmM}{r^2} e_r \Rightarrow \vec{F} = -\nabla U = \boxed{\frac{-GmM}{r^2} e_r = \vec{F}}$$

this is Newton's Universal Law of Gravitation.

E125 Consider the electric potential $V = \frac{-kQ}{r}$ then

$$\vec{E} = -\nabla V = \boxed{\frac{kQ}{r^2} e_r = \vec{E}} \text{ the Electric field of a static point charge } Q.$$

to get the force on q due to Q 's electric field we would multiply by q so $\vec{F} = \frac{kqQ}{r^2} e_r$ and the force is attractive if $qQ < 0$ (opposite polarity) or repulsive if $qQ > 0$ (like charges) in contrast to \vec{F}_{gravity} which has same $1/r^2$ dependence but is always attractive.

E126 Suppose $x^2 + y^2 + z^2 = R^2$ find the normal to this level surface. Notice this is $r^2 = R^2$ or $r = R$ (assuming $R > 0$) then $F(r, \theta, \phi) = r$ gives sphere as $F = R$. The normal is simply $\nabla F = e_r \frac{\partial r}{\partial r} + 0 + 0 = e_r$.

Remark: these examples are probably too simple to grasp the power of using spherical or cylindrical. Trust me they're very useful, and much easier to understand geometrically for a problem that has spherical or cylindrical symmetry.