

Functions of several variables

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A function is an assignment of an output for each input. The set of all allowed inputs is the domain, the set of all outputs is the range. We will use the terminology

" f is an apple-valued function of oranges"

this means f takes in an orange and outputs an apple.

Silly generalities aside we set apples = \mathbb{R} for now and notice oranges = \mathbb{R} (calculus I & II) or $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$ (calculus III).

Notation is $f: \mathbb{R}^m \rightarrow \mathbb{R}$ however usually we have

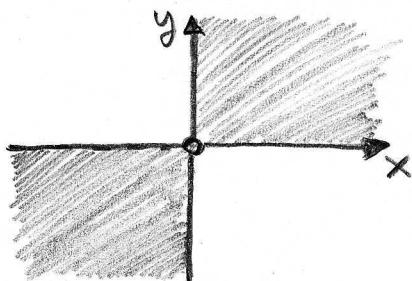
some $U \subset \mathbb{R}^m$ so $f: U \rightarrow \mathbb{R}$, then $\text{dom}(f) = U$.

Defn A function f of two variables is a rule that assigns each $(x, y) \in D \subseteq \mathbb{R}^2$ a unique real # $f(x, y)$.

$$\text{dom}(f) = D \quad \text{range}(f) = f(D) = \{f(x, y) \mid (x, y) \in D\}$$

All the same concerns as in precalc. enter here in the consideration of domains. Basically don't divide by zero and keep your square root inputs positive.

E18 $f(x, y) = \sqrt{xy} / (x^2 + y^2)$ find $\text{dom}(f)$. So we have to throw out the origin to avoid % by zero. Then we need $xy > 0 \Rightarrow$ either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$.



the $\text{dom}(f)$ consists of two disconnected parts.

GRAPHS of $f(x, y)$

to visualize $f(x)$ we need to look at $y = f(x)$ in \mathbb{R}^2 ,
to visualize $f(x, y)$ we need to look at $z = f(x, y)$ in \mathbb{R}^3 .

$$\text{graph}(f(x, y)) = \{(x, y, z) \mid z = f(x, y), (x, y) \in \text{dom}(f)\}$$

GRAPHS $z = f(x, y)$ and SURFACES in \mathbb{R}^3

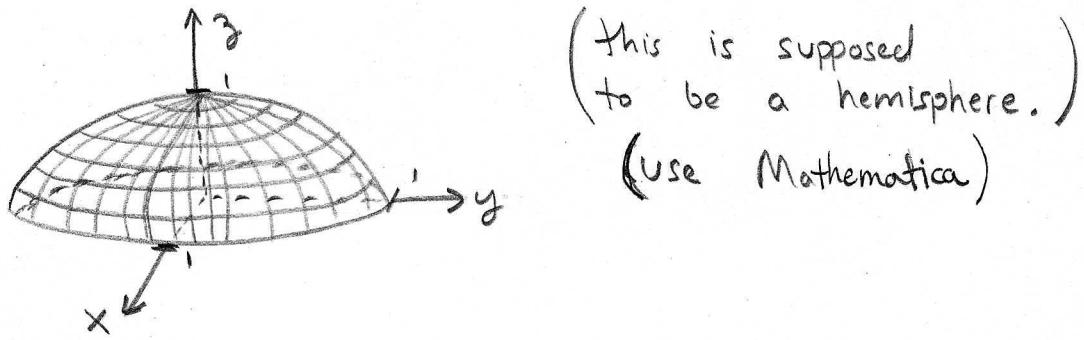
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Recall in \mathbb{R}^2 only shapes that passed the vertical line test could be interpreted as a graph $y = f(x)$, there were many other curves that were the collection of points in \mathbb{R}^2 satisfying some relation, usually algebraic. For example in \mathbb{R}^2 $x^2 + y^2 = 1$ is a circle and it cannot be seen as $y = f(x)$ for a single function (it takes two in fact $\pm \sqrt{1-x^2}$). These distinctions persist in \mathbb{R}^3 , surfaces of the form

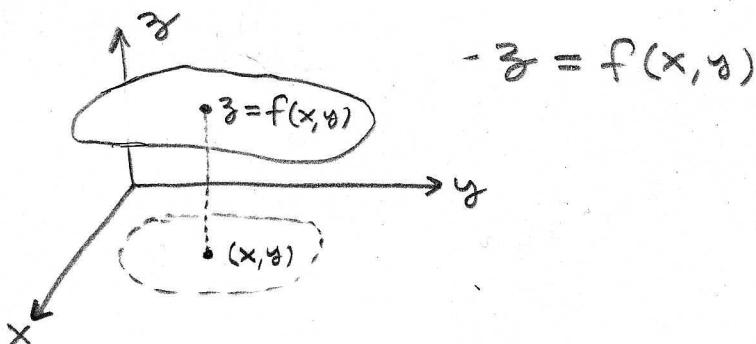
$$z = f(x, y)$$

are quite special. We refer to them as a "graph" in \mathbb{R}^3 . Stewart almost always finds a specialized formula for graphs, I'm much more interested that you understand the concept as opposed to blindly using Stewart's cook bookish formulas.

E19 $z = \sqrt{1 - x^2 - y^2}$ for $x^2 + y^2 \leq 1$.



E20



- I'm not going to attempt the pictures in the text. Please look over them to gain more intuition.

QUADRIC SURFACES : see table 1 of p. 844 for pretty pictures. (259)

Ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  Sphere if $a=b=c$

Elliptic Paraboloid : $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ 

Hyperbolic Paraboloid : $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ 

Cone : $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ 

Hyperboloid of One Sheet : $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ 

Hyperboloid of Two Sheets : $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 

Lets study these as an exercise in how to graph by hand. (Assume computers are evil, its the robot holocaust etc...)

E21 Ellipsoid. The name is appropriate, anyway we slice it we'll get an ellipse. I usually look at the coordinate planes then go from there to gather whatever other data might appear to be helpful.

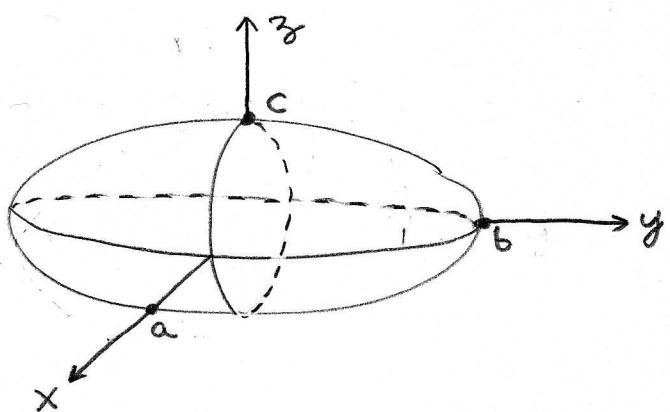
$$x=0 \Rightarrow \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$y=0 \Rightarrow \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$



$$z=0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Assuming $a, b, c > 0$ it looks something like this. Notice the idea is to use the cross-sections to get an idea where the surface is. This is essentially what Mathematica does for us.

E22) Cone: $z^2 = x^2 + y^2$

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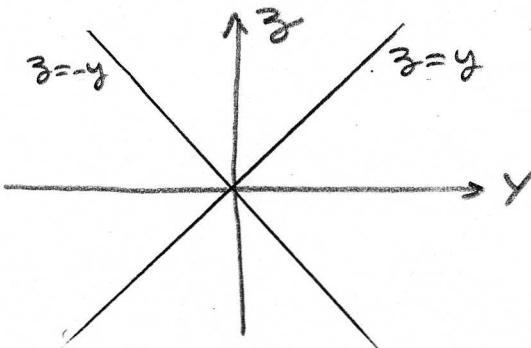
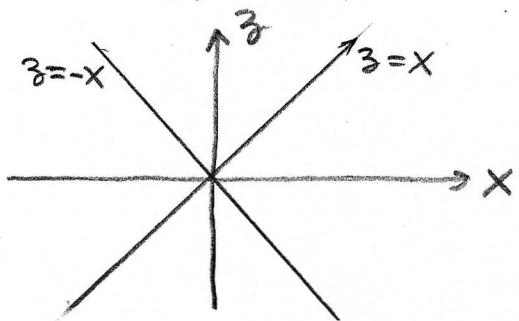
We can figure out somethings algebraically to begin,

$z=0 \Rightarrow x^2+y^2 \Rightarrow x=0$ and $y=0 \therefore$ intersects xy -plane only at the origin.

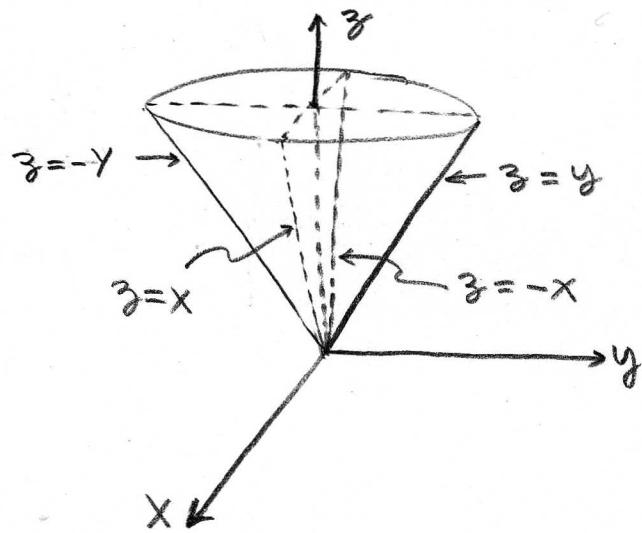
$$x=0 \Rightarrow z^2=y^2 \Rightarrow z=\pm y.$$

$$y=0 \Rightarrow z^2=x^2 \Rightarrow z=\pm x.$$

We can draw cross-sections of the surface with the coordinate planes $x=0$ and $y=0$



Then I'll attempt a 3-d rendition, (just for $z \geq 0$)



the trick is to draw the shape you imagine then check it with the cross-section lines (the text calls these "traces")

Remark: My point overall is that to plot a surface its helpful to look at particular slices then try to assemble the pieces. Certainly Mathematica is a big help here. I don't expect you to memorize the names of the surfaces (except the sphere I suppose that's common knowledge). but you should be able to find cross-sections by looking at coordinate planes and such. This may be important in setting up certain integrals later.

Def^b A rule that assigns $(x, y, z) \in D$ to $f(x, y, z) \in \mathbb{R}$ is a function $f : D \rightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^3$ is the domain of f , $\text{dom}(f) = D$. Also $\text{range}(f) = f(D) = \{f(x, y, z) \mid (x, y, z) \in D\}$.

- the definition for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ should likewise be the obvious assignment of $(x_1, x_2, \dots, x_n) \mapsto f(x_1, x_2, \dots, x_n) \in \mathbb{R}$.
- the graph $\{f(x, y, z)\} = \{(x, y, z, w) \mid w = f(x, y, z)\} \subseteq \mathbb{R}^4$ is kinda hard to visualize. We resort to level surfaces which are also helpful for visualizing $z = f(x, y)$ as it happens.

Def^b The level sets of $f : D \rightarrow \mathbb{R}$ for $D \subseteq \mathbb{R}^n$ are the collections of points in D such that $f(D) = k$ where k is some constant in the range of f .

n=2] level curve for $f(x, y)$ is $f(x, y) = k$ (a curve)

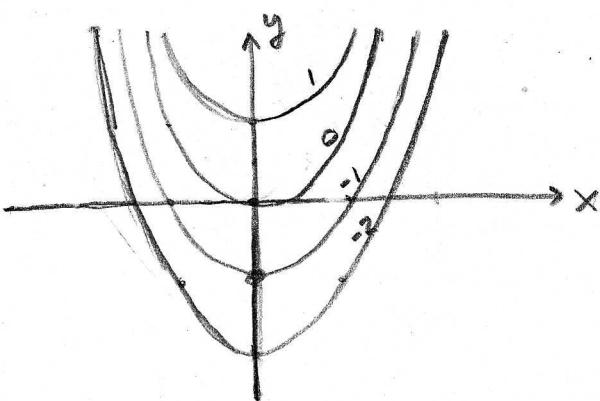
n=3] level surface for $f(x, y, z)$ is $f(x, y, z) = k$ (a surface)

n=4] level volume for $f(x, y, z, w)$ is $f(x, y, z, w) = k$ (a volume!)

E23 Consider $z = y - x^2 = f(x, y)$ we can sketch a few level curves in xy -plane. This is a "contour map"; you can

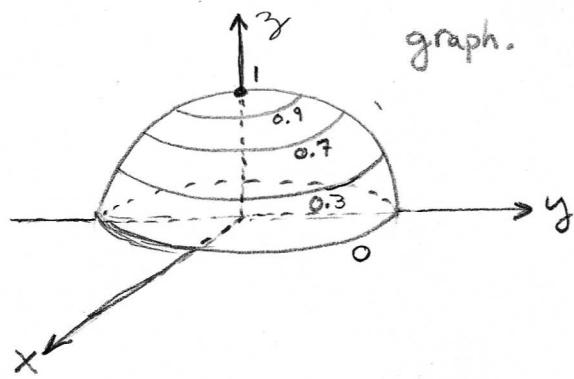
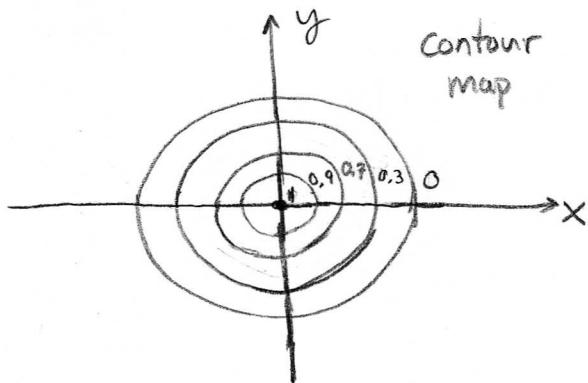
imagine $z = f(x, y)$ using the contours. The values $1, 0, -1, -2$ indicate that $f(x, y) = 1, 0, -1, 2$ respectively along those curves. My artistic ability is insufficient to sketch this one. ☺

Did I mention Mathematica can plot level curves (contour plot) along side a 3-d rendition of the graph.



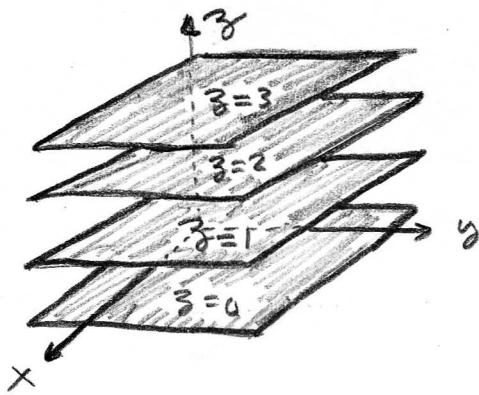
E24 Sketch a few contours for $z = \sqrt{1-x^2-y^2}$ then plot graph,

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this crude sketch gives us a pretty good idea about what the graph $z = \sqrt{1-x^2-y^2}$ looks like.

E25 $f(x, y, z) = z$ plot several level surfaces. The level surfaces are



$$z = k$$

this is the plane at $z=k$
which is parallel to the
 xy -plane.

Remark: It takes some practice to get comfortable viewing surfaces. Certainly Mathematica is a good tool to help you and it would be wise to use it as an aid. We've seen how algebraic relations implicitly or explicitly define some curve or surface in \mathbb{R}^3 . This is one description. The other description is the parametric one. You've seen that for curves, later we'll do parametrizations of surfaces. Both views are useful. Lastly I'd like to point out that cylindrical & spherical coordinates are a big conceptual aid in modelling surfaces, we delay discussion of that till later, closer to the time we'll use them most. If you wish to study them now and use them I wouldn't object.