

# (311)

## TANGENT PLANE TO $z = f(x, y)$ and LINEARIZATIONS of $f$ :

(We expand on §15.4 of Stewart, in essence.)

To begin we consider  $z = f(x, y)$ , we assume that  $f_x$  and  $f_y$  are continuous so we are assured that the tangent plane is well defined (see (306) for what happens otherwise).

PROPOSITION: The tangent plane to  $z = f(x, y)$  at  $(a, b, f(a, b))$  is

$$z = f(a, b) + \frac{\partial f}{\partial x} \Big|_{(a,b)} (x-a) + \frac{\partial f}{\partial y} \Big|_{(a,b)} (y-b)$$

provided that  $f$  is differentiable at  $(a, b)$ .

Pf: will have to wait until we give a better technical description of what a tangent plane is theoretically. Ignorance of that def<sup>n</sup> will not hinder us in our work on graphs. You may skip ahead to (318-319) for details, or see (306) for the def<sup>n</sup>.

Def<sup>n</sup>: Let  $f: S \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be differentiable at  $(a, b)$ . We say  $L: \mathbb{R}^2 \rightarrow \mathbb{R}$  is the linearization of  $f$  at  $(a, b)$  defined by

$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x} \Big|_{(a,b)} (x-a) + \frac{\partial f}{\partial y} \Big|_{(a,b)} (y-b)$$

we may add an  $f$  and  $(a, b)$  if we have several linearizations and need a distinguishing notation ( $L = L^f_{(a,b)}$  or  $L_f(a,b)$  perhaps)

Remark: the linearization  $L$  of  $f$  is the best linear approximation of the function near the base point of the linearization. This is the natural generalization of the tangent line approx. of a function, the closer to the point of tangency the closer the tangent line approximates the function.

**E71** Find the eq<sup>n</sup> of the tangent plane at  $(1, 2, 5)$  for  $f(x, y) = x^2 + y^2$ . We calculate  $f_x(1, 2)$  and  $f_y(1, 2)$ ,

$$f_x(1, 2) = 2x \Big|_{(1,2)} = 2 \quad f(1, 2) = 1^2 + 2^2 = 5$$

$$f_y(1, 2) = 2y \Big|_{(1,2)} = 4$$

Thus the tangent plane is  $\boxed{z = 5 + 2(x-1) + 4(y-2)}$

# TANGENT PLANES, LINEARIZATIONS, TOTAL DERIVATIVE

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**E72** Find the linearization of  $f(x,y) = x^2 + y^2$  at  $(1,2)$ . Then approximate  $f(2,2)$  and compare to the real-value. We found the tangent plane's eq<sup>2</sup> in **E71**) so we already know

$$L(x,y) = 5 + 2(x-1) + 4(y-2)$$

We approximate  $f$  via  $L$ ,

$$f(2,2) \approx L(2,2) = 5 + 2(2-1) + 4(0) = 7$$

Of course we can just evaluate  $f(2,2) = 2^2 + 2^2 = 8$  to see we have an absolute error of  $8-7 = 1$ . We can express these thoughts via the "increments" and "total differential"

$$\Delta z = f(x_2, y_2) - f(x_1, y_1)$$

$$\text{total differential} \rightarrow dz = f_x(x_1, y_1) \underbrace{(x_2 - x_1)}_{dx} + f_y(x_1, y_1) \underbrace{(y_2 - y_1)}_{dy}$$

In particular we have  $(x_1, y_1) = (1,2)$  and  $(x_2, y_2) = (2,2)$  thus  $dx = 1$  and  $dy = 0$  hence

$$dz = 2dx + 4dy = 2 + 0 = 2$$

$$\Delta z = f(2,2) - f(1,2) = 8 - 5 = 3$$

$$\text{Then } f(2,2) = f(1,2) + \Delta z = 5 + 3 = 8 \quad (\text{the true value})$$

$$L(2,2) = f(1,2) + dz = 5 + 2 = 7 \quad (\text{the approximate value})$$

Remark: this is a reasonable notation for approximation work, but I much prefer to use  $\Delta x = x_2 - x_1$  and  $\Delta y = y_2 - y_1$  for finite increments. Conceptually, when I write  $dx$  or  $dy$  then I have in mind an infinitesimal change in  $x$  or  $y$ . Stewart does not share my vision, see E4 on p. 933..

**Defn** The total differential of  $z = f(x,y)$  is defined to be

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df$$

Notice this is for  $f(x,y)$ . When we have  $w = f(x,y,z)$  then

$$\text{we will write } dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

(since  $w = f$ )

## Estimating Error with the Total Differential

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**E73** It is known that if we place resistors  $R_1$  and  $R_2$  in parallel then the effective resistance  $R$  of the system is

$$R = \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}$$

we can view  $R = f(R_1, R_2)$ , it is a function of two variables.

Now suppose  $R_1 = 10\Omega \pm 1\Omega$  and  $R_2 = 2\Omega \pm 0.5\Omega$ , here  $\Omega = \text{"ohm"}$  and we'll drop them for convenience sake. Interpretation:

$$dR_1 = 1 \quad \text{and} \quad dR_2 = 0.5$$

what is the order of our uncertainty in  $R$  then? Essentially upto a convention or two its the total differential in  $R$ ,

$$\begin{aligned} dR &= \frac{\partial R}{\partial R_1} \Big|_{(10,2)} dR_1 + \frac{\partial R}{\partial R_2} \Big|_{(10,2)} dR_2 \\ &= \left( \frac{-1}{(\frac{1}{R_1} + \frac{1}{R_2})^2} \right) \left( -\frac{1}{R_1^2} \right) \Big|_{(10,2)} dR_1 + \left( \frac{-1}{(\frac{1}{R_1} + \frac{1}{R_2})^2} \right) \left( -\frac{1}{R_2^2} \right) \Big|_{(10,2)} dR_2 \\ &= \frac{dR_1}{(1 + R_1/R_2)^2} \Big|_{(10,2)} + \frac{dR_2}{(R_2/R_1 + 1)^2} \Big|_{(10,2)} \\ &= \frac{1}{36} + \left( \frac{25}{36} \right) (0.5) = \frac{13.5}{36} = \boxed{0.375} \quad (\text{Mostly from } dR_2.) \end{aligned}$$

Remark: this is the largest uncertainty or error if you prefer. Be warned you should study error & measurements & statistics elsewhere.

**E74** Let  $R = R_1 + R_2 = g(R_1, R_2)$ . That is assume the resistors are in series this time. Find  $dR$  in this case,

$$\begin{aligned} dR &= \frac{\partial R}{\partial R_1} \Big|_{(10,2)} dR_1 + \frac{\partial R}{\partial R_2} \Big|_{(10,2)} dR_2 \\ &= dR_1 + dR_2 \\ &= 1 + 0.5 \\ &= \boxed{1.5 = dR} \quad (\text{Mostly from } dR_1) \end{aligned}$$

Remark: You can see the net-error is a consequence of both the error in the inputs and the eq<sup>n</sup> that gives the outputs. The total differential gives us an estimation of that net-error.