

Motion in Three Dimensions

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We have all the mathematics we need, now we give the path $\vec{r} : [a, b] \rightarrow C \subset \mathbb{R}^3$ where $t \mapsto \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ a physical interpretation. We suppose "t" is the time and consider a particular material object then for that object,

Defn/ $\vec{r}(t) \equiv$ the position at time $t = \langle x(t), y(t), z(t) \rangle$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} = \langle \dot{x}, \dot{y}, \dot{z} \rangle \equiv \text{velocity at time } t$$

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} \equiv \ddot{\vec{r}} = \langle \ddot{x}, \ddot{y}, \ddot{z} \rangle \equiv \text{acceleration at time } t$$

$$|\vec{v}(t)| \equiv |\vec{r}'(t)| = \frac{ds}{dt} = \text{speed at time } t = \dot{s}$$

these are determined by Newton's Laws. If \vec{F} is the total force which acts on the object of mass m then Newton's 2nd law states that in terms of momentum $\vec{P} \equiv m\vec{v}$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

Now usually the mass m is constant so we have the famous

$$\vec{F} = m\vec{a}$$

E43 Ignoring orbital effects and for m near the surface of the earth the force of gravity is approximated by

$\vec{F} = m\langle 0, 0, -g \rangle$ where $g = 9.8 \text{ m/s}^2$. We take the xy -plane to be horizontal and z to be vertical. Further suppose initially $\vec{r}(0) = \langle x_0, y_0, z_0 \rangle = \vec{r}_0$ and $\vec{v}(0) = \langle 0, 0, v_0 \rangle$.

$$\vec{F} = m\langle 0, 0, -g \rangle = m\langle \ddot{x}, \ddot{y}, \ddot{z} \rangle = m\langle \dot{v}_x, \dot{v}_y, \dot{v}_z \rangle = m\frac{d\vec{v}}{dt}$$

$$\Rightarrow \int_0^t \frac{d\vec{v}}{dt} dt = \int_0^t \langle 0, 0, -g \rangle dt$$

|| || motion is purely vertical.

$$\vec{v}(t) - \vec{v}(0) = \langle 0, 0, -gt \rangle \therefore \boxed{\vec{v}(t) = \langle 0, 0, v_0 - gt \rangle}$$

$$\frac{d\vec{r}}{dt} = \vec{v}(t) \Rightarrow \int_0^t \frac{d\vec{r}}{dt} dt = \int_0^t \langle 0, 0, v_0 - gt \rangle dt$$

$$\vec{r}(t) - \vec{r}(0) = \langle 0, 0, v_0 t - \frac{g}{2} t^2 \rangle \therefore \boxed{\vec{r}(t) = \langle x_0, y_0, z_0 + v_0 t - \frac{g}{2} t^2 \rangle}$$

TANGENTIAL AND NORMAL Components of \vec{V} and \vec{a}

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As we discussed the path instantaneously resides in the osculating plane. The velocity and acceleration are vectors in this plane and we can write \vec{v} and \vec{a} in terms of \vec{T} and \vec{N} . Also we attempt to further clarify the role of \dot{s} and \ddot{s} as they relate to the motion. As usual we assume the path is nonstop and nonlinear ($\vec{r} \neq 0$ and $\vec{r} \times \vec{r}' \neq 0 \forall t$)

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{v}(t)}{\dot{s}} \Rightarrow \boxed{\vec{v} = \dot{s} \vec{T}}$$

the velocity is purely tangential. Acceleration is not, consider,

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(\dot{s} \vec{T}\right) = \ddot{s} \vec{T} + \dot{s} \frac{d\vec{T}}{dt}$$

But we know that the derivative of \vec{T} relates to \vec{N} ,

$$\frac{d\vec{T}}{ds} = \kappa \vec{N} \quad \text{and} \quad \frac{d\vec{T}}{dt} = \frac{ds}{dt} \frac{d\vec{T}}{ds} \Rightarrow \underbrace{\frac{d\vec{T}}{dt}}_{\dot{s}} = \dot{s} \kappa \vec{N}$$

Therefore we find,

$$\boxed{\vec{a} = \ddot{s} \vec{T} + \dot{s}^2 \kappa \vec{N}}$$

Remark: the magnitude of the acceleration is $|a| = \sqrt{(\ddot{s})^2 + (\dot{s}^2 \kappa)^2}$.

intuitively you might be tempted to suppose $|a| = \ddot{s} = \frac{d^2 s}{dt^2}$ but this is not the case due to the Normal component.

When $\kappa = 0$ (a case we must treat independently because the Frenet - Serret formulas apply to nonlinear nonstop paths)

one has only the tangential component and $|a| = \ddot{s}$ (special case.)

Components:

To find the \vec{T} or \vec{N} components we simply take the dot-product with \vec{T} or \vec{N} (these are orthogonal $\vec{T} \cdot \vec{N} = 0$)

$$a_T = \vec{a} \cdot \vec{T} = \ddot{s}$$

$$a_N = \vec{a} \cdot \vec{N} = \dot{s}^2 \kappa$$

CONSTANT SPEED CIRCULAR MOTION

Suppose that an object travels in a circle at a constant angular velocity. In particular suppose after time T (period) the object goes a full cycle around a circle of radius R ; that is $V_0 \equiv 2\pi R/T = \frac{ds}{dt}$. The parametrization fitting this description is the following if we assume $\vec{r}(0) = (R, 0, 0)$.

$$\vec{r}(t) = \langle R \cos(2\pi t/T), R \sin(2\pi t/T), 0 \rangle$$

$$\vec{r}'(t) = \frac{2\pi R}{T} \langle -\sin(\omega t), \cos(\omega t), 0 \rangle : \text{letting } \omega = \frac{2\pi}{T}$$

$$|\vec{r}'(t)| = \frac{2\pi R}{T} = V_0 = \omega R$$

$$\hat{\vec{r}}(t) = \langle -\sin(\omega t), \cos(\omega t), 0 \rangle$$

$$\hat{\vec{r}}'(t) = \omega \langle -\cos(\omega t), -\sin(\omega t), 0 \rangle$$

$$|\hat{\vec{r}}'(t)| = \omega$$

$$\hat{\vec{N}}(t) = \frac{\hat{\vec{r}}'(t)}{|\hat{\vec{r}}'(t)|} = \langle -\cos(\omega t), -\sin(\omega t), 0 \rangle$$

Lets collect our thoughts,

$$\vec{v} = V_0 \hat{\vec{r}}$$

$$\vec{a} = \vec{r}''(t) = \omega^2 R \langle -\cos(\omega t), -\sin(\omega t) \rangle = \omega^2 R \hat{\vec{N}}$$

Compare this to our general result

$$\vec{a} = \ddot{s} \hat{\vec{r}} + (\dot{s})^2 \hat{\vec{N}} = \omega^2 R \hat{\vec{N}} = \frac{V_0^2}{R^2} R \hat{\vec{N}} = \frac{V_0^2}{R} \hat{\vec{N}}$$

Therefore for any force that results in circular motion of a constant rate we must have

$$\boxed{\vec{F} = m\vec{a} = \frac{mV_0^2}{R} \hat{\vec{N}} = \frac{mV_0^2}{R} (-\hat{\vec{r}})}$$

Remark: this is a very special case. Generally there is also $\vec{a}_T \neq 0$. We note that to obtain circular motion of constant ω we need that $\vec{F} \cdot \hat{\vec{r}} = 0$, probably the magnetic force

$\vec{F} = q\vec{v} \times \vec{B}$ is the most famous case. It turns out that even in the relativistic case it still causes circular motion. I'll show you if you ask.

E44 Let $\vec{r}(t) = \langle t, 2t, t^2 \rangle$ find a_T and a_N . Calculate,

$$\vec{r}'(t) = \langle 1, 2, 2t \rangle \Rightarrow \dot{s} = |\vec{r}'(t)| = \sqrt{5 + 4t^2}$$

Now use our result from (281).

$$a_T = \ddot{s} = \frac{d}{dt}(\sqrt{5+4t^2}) = \boxed{\frac{4t}{\sqrt{5+4t^2}} = a_T}$$

To find a_N notice $|\vec{a}|^2 = |a_T|^2 + |a_N|^2$ it's easy to calculate $\vec{a} = \vec{r}''(t) = \langle 0, 0, 2 \rangle$ thus $|\vec{a}|^2 = 4$. So,

$$\begin{aligned} a_N &= \sqrt{|\vec{a}|^2 - |a_T|^2} \\ &= \sqrt{4 - \frac{16t^2}{5+4t^2}} \\ &= \sqrt{\frac{20 + 16t^2 - 16t^2}{5+4t^2}} \\ &= \boxed{2\sqrt{5}/\sqrt{5+4t^2} = a_N} \end{aligned}$$

Remark: if we knew κ then we could have used $a_N = \kappa \dot{s}^2$. On the other hand now we can find $\kappa = \frac{a_N}{\dot{s}^2}$

E45 Lets find a_T and a_N for the circle we studied explicitly on (282). We found that $\dot{s} = v_0$. Also we have calculated that $\kappa = 1/R$ for a circle of radius R in view of E37 on (275), notice we know the result holds despite the different parametrization on (282). The curvature is an intrinsic path independent property of the circle. Then using the results of (281),

$$a_T = \ddot{s} \vec{T} = 0$$

$$a_N = \dot{s}^2 \kappa \vec{N} = \frac{v_0^2}{R} \vec{N}$$

$$\therefore \vec{a} = a_T \vec{T} + a_N \vec{N} = \boxed{\frac{v_0^2}{R} \vec{N} = \vec{a}}$$

Remark: there are many shortcuts formulas. I don't require you to know them, if you use them then you should derive them on the test (not huk though)