

PARTIAL DIFFERENTIATION

For a function of two independent variables x and y we define,

Defⁿ/ The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is (provided the limit below exists)

$$\frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \frac{d}{dx} [f(x, y_0)] \Big|_{x=x_0} = \lim_{h \rightarrow 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

also we consider the partial derivative w.r.t. x as a function in its own right with values given in the obvious way.

$$\frac{\partial f}{\partial x}(x, y) \equiv \frac{\partial f}{\partial x} \Big|_{(x, y)} = f_x = z_x = \frac{\partial z}{\partial x}$$

where the last two notations are appropriate when considering $z = f(x, y)$.

Care to guess what $\frac{\partial f}{\partial y}$ means? It's the same,

Defⁿ/ We define the partial derivative w.r.t y at (x_0, y_0) to be

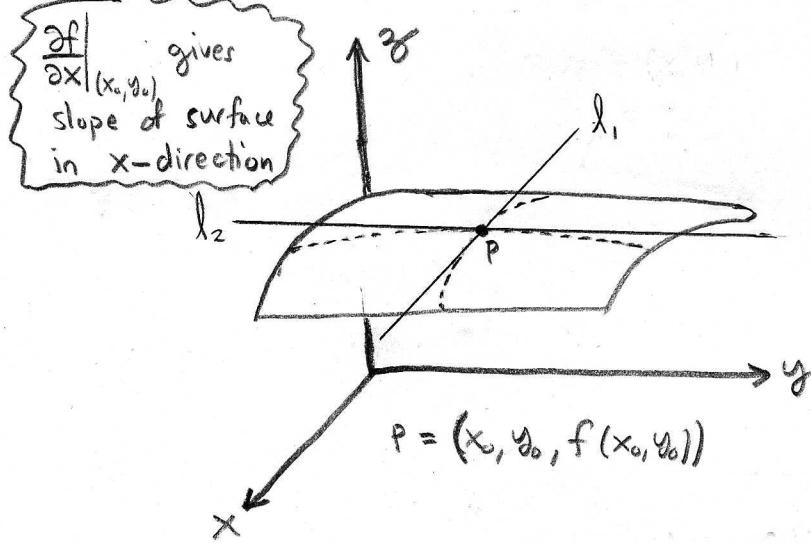
$$\frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} = \frac{d}{dy} [f(x_0, y)] \Big|_{y=y_0} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

We also consider the partial derivative w.r.t. y as a function in its own right, the values given pointwise by what we just defⁿ,

$$\frac{\partial f}{\partial y}(x, y) \equiv \frac{\partial f}{\partial y} \Big|_{(x, y)} = f_y = z_y = \frac{\partial z}{\partial y}$$

where the last two are appropriate for $z = f(x, y)$.

Partial Derivatives have a nice geometric meaning. The dotted lines indicate



You could consider them functions of one variable on $y = y_0$ and $x = x_0$. In other words the dotted lines are the intersection curves of $z = f(x, y)$ with $x = x_0$ & $y = y_0$. The lines l_1 & l_2 lift off the surface are tangent to those curves.

If the eqⁿ $z = f(x, y)$ was a plane then l_1 and l_2 would actually reside on the plane. Geometrically we say the plane that passes through $(x_0, y_0, f(x_0, y_0))$ and just kisses $z = f(x, y)$ is the tangent plane. We'll find a more technical description on 306. Essentially the tangent plane is the best linear approx to $z = f(x, y)$.

Remark: A function $f(x, y)$ is differentiable at (a, b) if it has a unambiguous tangent plane at $(a, b, f(a, b))$. There are functions for which f_x and f_y exist at (a, b) yet there is no tangent plane. It turns out we need f_x & f_y to be continuous at (a, b) to insure differentiability of f . We denote all functions diff. at (a, b) by $C^1(a, b)$, for more skip to (306). For now we consider the basics,

E49 $F(x, y) = x^2 + y^2$.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} [x^2 + y^2] = \frac{\partial}{\partial x} [x^2] + \frac{\partial}{\partial x} [y^2]^0 = [2x] \quad : y \text{ is } \underline{\text{constant}} \text{ with respect to } x$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} [x^2 + y^2] = [2y] \quad : \text{we regard } x \text{ as constant as we perform the } \% \text{ operation.}$$

E50 $F(x, y) = xe^{xy}$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} (xe^{xy}) = \frac{\partial x}{\partial x} e^{xy} + x \frac{\partial}{\partial x} (e^{xy}) \quad : \text{product rule}$$

$$F_x = e^{xy} + xy e^{xy} \quad : \text{chain rule, remember } y \text{ is regarded constant}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} (xe^{xy}) = x \frac{\partial}{\partial y} (e^{xy}) = x e^{xy} \frac{\partial}{\partial y} (xy) = [x^2 e^{xy}] = F_y$$

here I wrote out the chain rule, not always need, but may help in messy cases.

E51 $\bar{z}^2 = \sin(xy) + x + \ln(y)$. Suppose that x & y are independent and \bar{z} is dependent ; $\bar{z} = \bar{z}(x, y)$. We use implicit differentiation to find implicit formulas for \bar{z}_x and \bar{z}_y .

$$\frac{\partial}{\partial x} [\bar{z}^2] = 2\bar{z} \frac{\partial \bar{z}}{\partial x}$$

$$\frac{\partial}{\partial x} [\sin(xy) + x + \ln(y)] = y \cos(xy) + 1$$

But these are equal so likewise we calculate,

$$\frac{\partial \bar{z}}{\partial x} = \frac{1}{2\bar{z}} [y \cos(xy) + 1]$$

question: why is this implicit?

$$2\bar{z} \frac{\partial \bar{z}}{\partial y} = x \cos(xy) + \frac{1}{y} \quad : \quad$$

$$\frac{\partial \bar{z}}{\partial y} = \frac{1}{2\bar{z}} [x \cos(xy) + \frac{1}{y}]$$

PARTIAL DERIVATIVES OF $f(x_1, x_2, \dots, x_n)$:

The defⁿ of $\frac{\partial f}{\partial x_k}$ is essentially the same as that for $f(x, y)$, the meaning is / that we take the ordinary derivative w.r.t. x_k while holding all the other variables fixed. That is,

$$\frac{\partial f}{\partial x_k}(x_1, x_2, \dots, x_n) = \lim_{h \rightarrow 0} \frac{f(x_1, x_2, \dots, x_k+h, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h}$$

We also may employ the notations,

$$\frac{\partial f}{\partial x_k} = \partial_k f = f_{x_k}$$

E52 Let $g(x, y, z) = xy^2z^3 + \sin(xy\bar{z})$ then

$$g_x = y^2z^3 + yz \cos(xy\bar{z}) : y, z \text{ treated as constants.}$$

$$g_y = 2xyz^3 + xz \cos(xy\bar{z}) : x, z \text{ treated as constants.}$$

$$g_z = 3xy^2z^2 + xy \cos(xy\bar{z}) : x, y \text{ treated as constants.}$$

E53 Suppose $r = \sqrt{x^2 + y^2 + z^2}$.

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \frac{\partial}{\partial x} [x^2+y^2+z^2] = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Likewise $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = z/r$.

Remark: the hawk solⁿ has many more examples.

HIGHER PARTIAL DERIVATIVES:

have the obvious meaning, we simply iterate. For example,

E54 Let $f(x, y) = xy^2$.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial x} \right] = \frac{\partial}{\partial x} [y^2] = 0.$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = \frac{\partial}{\partial x} [2xy] = 2y.$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial x} \right] = \frac{\partial}{\partial y} [y^2] = 2y.$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial y} \right] = \frac{\partial}{\partial y} [2yx] = 2x.$$

interesting, is it always the case that $f_{xy} = f_{yx}$?

- you may consult your hawk for added examples on this topic.

A CURIOUS EXAMPLE: Why $f_{xy} \neq f_{yx}$ ALWAYS.

(295)

$$f(x,y) = \begin{cases} (x^3y - xy^3)/(x^2 + y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

When $(x,y) \neq (0,0)$ it's a simple matter to differentiate,

$$f_x = \frac{(3x^2y - y^3)(x^2 + y^2) - 2x(x^3y - xy^3)}{(x^2 + y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$$

$$f_y = \frac{x^5 - 4y^2x^3 - y^4x}{(x^2 + y^2)^2}$$

$$f_{xy} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} = f_{yx}(x,y) \text{ for } (x,y) \neq 0.$$

At the origin we need to use the defⁿ of partial differentiation,

$$f_x(0,0) = \lim_{h \rightarrow 0} \left[\frac{f(h,0) - f(0,0)}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{0 - 0}{h} \right] = 0.$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \left[\frac{f(0,h) - f(0,0)}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{0 - 0}{h} \right] = 0.$$

$$f_{xy}(0,0) \equiv \frac{\partial f_x}{\partial y}(0,0) = \lim_{h \rightarrow 0} \left[\frac{f_x(0,h) - f_x(0,0)}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{-h^5/(h^2)^2 - 0}{h} \right] = -1.$$

$$f_{yx}(0,0) \equiv \frac{\partial f_y}{\partial x}(0,0) = \lim_{h \rightarrow 0} \left[\frac{f_y(h,0) - f_y(0,0)}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{h^5/h^4 - 0}{h} \right] = 1.$$

Therefore $f_{xy} \neq f_{yx}$ since at $(0,0)$ they disagree. You might object that this is picky on our part, well sorry it's math. The trouble here is that f_{xy} is not continuous at $(0,0)$, everywhere else it is and in all those places $f_{xy}(x,y) = f_{yx}(x,y) \forall (x,y) \neq (0,0)$.

CLAIRAUT'S THⁿ: Suppose f is defined on some disk containing (a,b) . If the functions f_{xy} and f_{yx} are both continuous on D then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Proof: you can look it up in the appendix, or find a more advanced text perhaps. Anyway you can see from the counterexample given above that it takes a fairly contrived function to escape the usual fact that $f_{xy} = f_{yx}$.

Remark: we have covered §15.2 and §15.3 approximately on pgs. (290) → (295). Next we discuss the chain rule for several variables, after that I include some material on constrained partials (seemingly not in Stewart) then we will study the tangent plane and linearization (§15.4 + §15.6).