

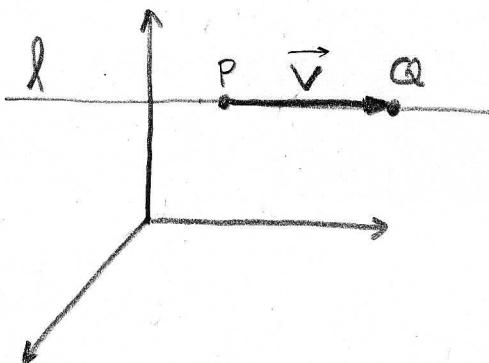
# LINES & PLANES IN 3-d

(251)

To begin we introduce the notion of a parametrized line.

The parameter "t" or "s" (or whatever you prefer) is an extra variable introduced to give a convenient description of the line. The parameter could be chosen to be arclength or perhaps  $x, y$  or  $z$ . There is much freedom in the choice.

The standard trick: well it's not much of a trick really, but to parametrize a line  $\ell$  that passes through points  $P$  and  $Q$  we construct  $\vec{v} = Q - P$  then write (here  $P = \vec{P}$ )  $\vec{r}(t) = \vec{P} + t\vec{v}$



this has  $\vec{r}(0) = P$  and  $\vec{r}(1) = Q$ .

Notice the parametrization gives the line  $\ell$  a direction, we can say  $\ell$  is oriented if we insist on a direction for its parametrization. Otherwise, there are two choices ( $\rightarrow$ ) or ( $\leftarrow$ ).

Def<sup>b</sup>/ Suppose that  $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$  and  $\vec{v} = \langle a, b, c \rangle$  then  $\vec{r}(t) = \vec{r}_0 + t\vec{v}$  is a line with initial point  $\vec{r}_0$  and direction  $\vec{v}$ . The parametric eq<sup>n</sup>'s for  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$  are simply found by looking at  $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ ,

$$x(t) = x_0 + at$$

$$y(t) = y_0 + bt$$

$$z(t) = z_0 + ct \quad (\text{note } a, b, c \text{ are "direction #'s" for the line})$$

E10 find parametric eq<sup>n</sup>'s of line with direction  $\langle 1, 0, 1 \rangle = \vec{v}$  and initial point  $\vec{r}_0 = \langle \pi, \pi, \pi \rangle$ . Well  $r(t) = r_0 + tv$  so  $\vec{r}(t) = \langle \pi, \pi, \pi \rangle + t\langle 1, 0, 1 \rangle = \langle \pi+t, \pi, \pi+t \rangle$

$x = \pi + t$
$y = \pi$
$z = \pi + t$

Def<sup>n</sup>/ If  $L$  is a line in the  $\vec{V} = \langle a, b, c \rangle$

direction that passes through  $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$  then  
the symmetric eq<sup>n</sup>'s for  $L$  are  $(a, b, c \neq 0)$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

pragmatically I don't use this formula, rather I usually just solve the parametric eq<sup>n</sup>'s for  $t$ .

E11 Suppose  $L: \vec{F}(t) = \langle 3-t, t+5, 2t+8 \rangle$  then

$$\begin{aligned} x = 3-t &\Rightarrow t = 3-x \\ y = t+5 &\Rightarrow t = 5-y \\ z = 2t+8 &\Rightarrow t = \frac{1}{2}(z-8) \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow t = \boxed{3-x = 5-y = \frac{1}{2}(z-8)}$$

Ok, to be more picky we should write  $\frac{x-3}{-1} = \frac{y-5}{-1} = \frac{z-8}{2}$   
then we can identify  $\vec{V} = \langle -1, -1, 2 \rangle$  and  $\vec{r}_0 = \langle 3, 5, 8 \rangle$   
of course those facts are obvious from  $\vec{F}(t)$  to begin.

E12 In E10 we had  $\vec{V} = \langle 1, 0, 1 \rangle$  so  $b = 0$ , clearly  
we cannot divide by  $b$ ! So the sort-of symmetric eq<sup>n</sup>'s are

$$t = \boxed{x-\pi = 3-\pi \quad \text{and} \quad y = \pi}$$

Remark: the nice thing about parametric eq<sup>n</sup>'s for a line  
is that we always can write  $\vec{F}(t) = \vec{r}_0 + t\vec{V}$ , as opposed  
to the phenomenon we encounter in E12 showing the symmetric  
eq<sup>n</sup>'s can only be written for a certain subclass of all lines.  
I will by default use parametric description for our lines. Also  
later when we consider motion the  $t$  will be identified  
as time and  $\vec{F}(t)$  has a nice physical interpretation.

LINE SEGMENTS: are easy. You just restrict the domain of  
 $t$  so that it cuts-off the rest of the line. The  
line segment from  $\vec{r}_0 = (x_0, y_0, z_0)$  to  $\vec{r}_1 = (x_1, y_1, z_1)$  is

$$\vec{F}(t) = (x_0, y_0, z_0) + t(x_1 - x_0, y_1 - y_0, z_1 - z_0) \quad 0 \leq t \leq 1$$

Check it out,  $\vec{F}(0) = \vec{r}_0$  and  $\vec{F}(1) = \vec{r}_1$ . Again the  
parametrization gives it an orientation  $\vec{r}_0 \rightarrow \vec{r}_1$  (arrow along  
increasing  $t$ )

## Parallel & Skew Lines

Let  $L_1$  go in the  $\vec{V}_1$ -direction and  $L_2$  in the  $\vec{V}_2$ -direction.

Then lines  $L_1$  &  $L_2$  are parallel iff  $\vec{V}_1 = k\vec{V}_2$  for  $k \neq 0$ . The lines  $L_1$  &  $L_2$  are skew iff they are not parallel.

**E13** Show  $L_1 : \vec{r}_1(t) = \langle 3+t, 2+t, 1+t \rangle$  and  $\vec{r}_2(t) = \langle 3t, 3t, 3t+6 \rangle$  are parallel. Well notice

$$\vec{r}_1(t) = \langle 3, 2, 1 \rangle + t \langle 1, 1, 1 \rangle = \vec{r}_1(0) + t\vec{V}_1$$

$$\vec{r}_2(t) = \langle 0, 0, 6 \rangle + t \langle 3, 3, 3 \rangle = \vec{r}_2(0) + t\vec{V}_2$$

Clearly  $\vec{V}_2 = 3\vec{V}_1$ , thus  $L_1$  &  $L_2$  are parallel.

**E14** Show  $L_1 : \vec{r}_1(t) = \langle 1+t, 3t-2, 4-t \rangle$  and  $L_2 : \vec{r}_2(t) = \langle 2t, 3+t, 4t-3 \rangle$  do not intersect. To be fair we should not check if  $\exists t$  such that  $\vec{r}_1(t) = \vec{r}_2(t)$ , because they might intersect at different  $t$ . So introduce  $s$  for  $\vec{r}_2$ . Examine  $\vec{r}_1(t) = \vec{r}_2(s)$ . This gives

$$\begin{aligned} 1+t &= 2s \\ 3t-2 &= 3+s \\ 4-t &= 4s-3 \end{aligned} \quad \begin{array}{l} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \quad \begin{aligned} 3t &= 6s-3 = 3+s+5 \Rightarrow 5s = 11 \therefore s = \underline{\underline{11/5}} \\ 3t &= 12-12s+9 = 3+s+5 \Rightarrow 13s = 13 \therefore s = \underline{\underline{1}} \end{aligned}$$

Therefore since  $1 \neq 11/5$  it is seen these eq's have no sol<sup>n</sup> hence  $\nexists s, t$  such that  $\vec{r}_1(t) = \vec{r}_2(s) \therefore L_1$  &  $L_2$  do not intersect.

Moreover, these lines are skew since  $L_1$  has direction vector  $\langle 1, 3, -1 \rangle$  whereas  $L_2$  has direction vector  $\langle 2, 1, 4 \rangle$ . It is simple to show there does not exist a  $k \in \mathbb{R}$  such that  $\langle 1, 3, -1 \rangle = k\langle 2, 1, 4 \rangle$ . (Thus  $L_1$  and  $L_2$  point in different directions)

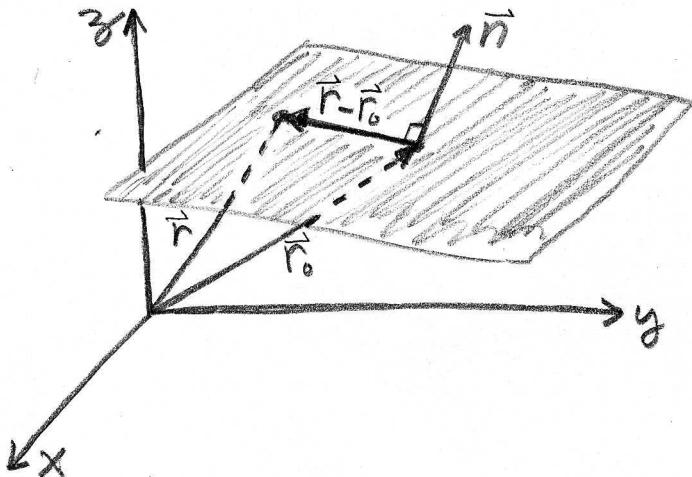
# PLANES IN $\mathbb{R}^3$

A plane is specified by a point  $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$  and a vector  $\vec{n} = \langle a, b, c \rangle$  which is orthogonal to the plane. The plane is defined to be the set of all points  $\vec{r} = (x, y, z) \in \mathbb{R}^3$  such that

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

vector eq<sup>n</sup> of plane

the vector  $\vec{r} - \vec{r}_0$  lies in the plane, I'll try to draw it



the plane is not finite, I just draw it that way. It goes on and on.

Now if  $\vec{n} = \langle a, b, c \rangle$  then the plane is given by

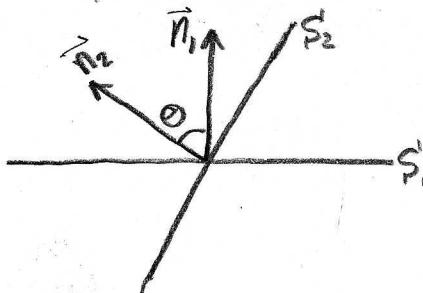
$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$= a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- the scalar eq<sup>n</sup>'s for the plane, through  $(x_0, y_0, z_0)$  with normal  $\langle a, b, c \rangle$ .

We say two planes are parallel if their normal vectors  $\vec{n}_1$  and  $\vec{n}_2$  are parallel, otherwise the angle between the normal vectors defines the angle between the planes. (Your book says acute angle but that seems impossible w/o reddefining certain normals...)

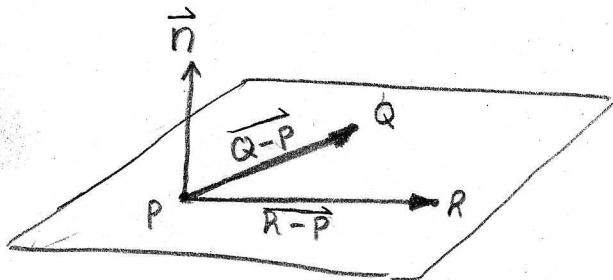
(sideview)



$S_1$  a plane with normal  $\vec{n}_1$ ,  
 $S_2$  a plane with normal  $\vec{n}_2$

$$\Theta = \text{angle between planes} = \cos^{-1} \left( \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right)$$

**E15** Find eq's of plane possessing points  $(0, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 1, 1)$ . Let me draw a picture, define



$$P = (0, 0, 0)$$

$$Q = (0, 1, 0) = \hat{j}$$

$$R = (1, 1, 1) = \hat{i} + \hat{j} + \hat{k}$$

$$Q - P = Q.$$

$$R - P = R.$$

$$\vec{n} = (Q - P) \times (R - P) = \hat{j} \times (\hat{i} + \hat{j} + \hat{k}) = \hat{j} \times \hat{i} + \hat{j} \times \hat{k} = \langle 1, 0, -1 \rangle.$$

Thus the eq's of the plane are (using  $\vec{P} = \vec{r}_0$ )  $x - z = 0$

Remark: remember the hwh sol<sup>b</sup> has master problems.

**E16** Find plane through  $(1, 2, 3)$  with normal parallel to the intersection line of the planes

$$x + y + z = 10 \quad \text{and} \quad 2x + 3y + z = 20.$$

General Principle: intersection is where both eq's are true, to quantify it pick something in common and equate, here  $z$  is a natural choice,

$$z = 10 - x - y = 20 - 2x - 3y$$

$$\Rightarrow 2y + x = 20 - 10 \Rightarrow x = 10 - 2y.$$

So we can parametrize the line by the  $y$ -coordinate,

$$\begin{aligned} \vec{r}(y) &= \langle 10 - 2y, y, 10 - x - y \rangle, x = 10 - 2y \\ &= \langle 10 - 2y, y, y \rangle \\ &= \langle 10, 0, 0 \rangle + y \langle -2, 1, 1 \rangle. \end{aligned}$$

The direction of the line of intersection is  $\langle -2, 1, 1 \rangle$  our plane is thus

$$-2(x-1) + (y-2) + (z-3) = 0$$

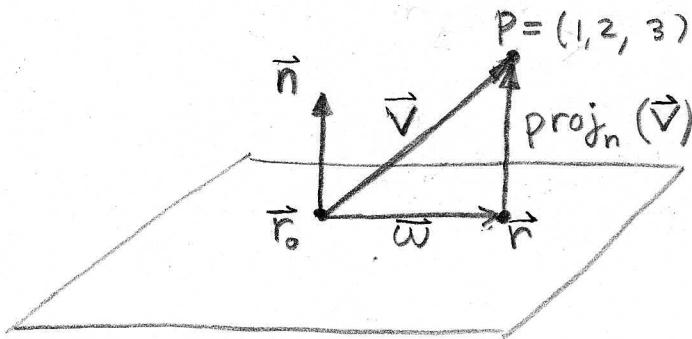
Remark: there is a shorter more efficient sol<sup>b</sup>. Just pick two points on line, don't try to find  $\vec{r}(y)$ .

Remark: there are many convoluted ways to give you information to find the eq's of a plane. What I always do is look for a point and some way to get the normal. Your homework explores various set-ups. You should try to understand the concept, remembering a dozen different formulas is not the best way.

E17 I return to the question at the bottom of 239, If we have the plane  $x - y + 10z = 10$  then what point on plane is closest to  $P = (1, 2, 3)$ ? Lets pick a point on the plane. Choose,

$$x = 0, y = 0 \Rightarrow 10z = 10 \therefore z = 1 \Rightarrow \vec{r}_0 = (0, 0, 1)$$

the normal to the plane is  $\vec{n} = \langle 1, -1, 10 \rangle$  so we can draw a schematic picture of what we have and want



$$\vec{v} = P - r_0 = \langle 1, 2, 2 \rangle$$

$\vec{r}$  = the desired point

$$\vec{r} = \vec{r}_0 + \vec{w} = \vec{P} - \text{proj}_{\vec{n}}(\vec{v})$$

$$\text{notice } |\vec{n}| = \sqrt{102}$$

$$\begin{aligned}\vec{r} &= \vec{P} - \text{proj}_{\vec{n}}(\vec{v}) \\ &= (1, 2, 3) - \frac{\vec{n} \cdot \vec{v}}{|\vec{n}|^2} \vec{n} \\ &= \langle 1, 2, 3 \rangle - \frac{1}{102} (\langle 1, -1, 10 \rangle \cdot \langle 1, 2, 2 \rangle) \langle 1, -1, 10 \rangle \\ &= \langle 1, 2, 3 \rangle - \left(\frac{1}{102}\right) \langle 1, -1, 10 \rangle \\ &= \frac{1}{102} \langle 102 - 1, 204 + 1, 306 - 10 \rangle \\ &= \frac{1}{102} \langle 83, 223, 116 \rangle \therefore \boxed{\left(\frac{83}{102}, \frac{223}{102}, \frac{116}{102}\right)} \text{ closest point.}\end{aligned}$$

As a check is  $\vec{r}$  on the plane?  $\frac{83}{102} - \frac{223}{102} + \frac{116}{102} = \frac{1020}{102} = 10$ , yep.

### PROJECTION ONTO PLANE WITH NORMAL $n$

The shadow of the vector  $\vec{v}$  onto the plane is obtained by subtracting the component that is off the plane, namely  $\text{proj}_{\vec{n}}(\vec{v})$

$$\pi_{\text{plane}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{n}}(\vec{v})$$

assuming  $\vec{v}$  has its tail on the plane. Its nice if we can put  $\vec{v}$  at the origin, otherwise we have to pick a point  $\vec{r}_0$  etc..., using  $\vec{r}_0 = 0$  is preferred.