Einstein said his greatest contribution to physics was this notation. I'm not sure I agree, but here it is:

- When an index is repeated we sum over all its values.

Let's examine how this convention helps us write a vector $\vec{A} =< A_1, A_2, A_3 >$ as $\vec{A} = A_k e_k$. We denote the xyz-coordinate unit vectors by $e_i$ for $i = 1, 2, 3$. In particular,

$$\boxed{e_1 =< 1,0,0 > \qquad e_2 =< 0,1,0 > \qquad e_3 =< 0,0,1 > .}$$

Observe that,

$$\begin{aligned} \vec{A} =< A_1, A_2, A_3 > &=< A_1, 0, 0 > + < 0, A_2, 0 > + < 0, 0, A_3 > \\ &= A_1 < 1,0,0 > + A_2 < 0,1,0 > + A_3 < 0,0,1 > \\ &= A_1 e_1 + A_2 e_2 + A_3 e_3 \\ &= A_k e_k \leftarrow \text{repeated index notation here} \end{aligned}$$

Geometrically this idea is actually very important. This is the algebraic statement that a vector can be written as a sum of its vector components. We use this a lot in problems in physics, we like to break down a vector into its components because then you can calculate lots of things easily.

Perhaps you have not thought about equality of vectors in component notation:

$$\boxed{\vec{A} = \vec{B} \quad \Longleftrightarrow \quad \vec{A} = A_k e_k, \ \vec{B} = B_k e_k \text{ and } A_k = B_k \text{ for each } k = 1, 2, 3.}$$

You know this already, vectors are equal only if all the components match.

***Example 1:*** The Dot Product: let $\vec{A} =< A_1, A_2, A_3 > \text{ and } \vec{B} =< B_1, B_2, B_3 >$ then,

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = A_m B_m$$

***Example 2:*** A simple proof that $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. I'll give the proof brute-force and in index-notation just to contrast. Both are easy here: ( I use $\equiv$ to indicate I am applying the definition directly)

1. (brute-force) $\vec{A} \cdot \vec{B} \equiv A_1 B_1 + A_2 B_2 + A_3 B_3 = B_1 A_1 + B_2 A_2 + B_3 A_3 \equiv \vec{B} \cdot \vec{A}$
2. (Einstein) $\vec{A} \cdot \vec{B} \equiv A_k B_k = B_k A_k \equiv \vec{B} \cdot \vec{A}$.

The beauty of breaking a vector calculation down to the level of components is that components are numbers so you can apply ordinary arithmetic to them. See, for each $k$ the object $A_k$ is just some number.

***Example 3:*** Prove that $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$. Notice that vector addition is defined so that the components of the sum of two vectors is the sum of the components; $(B+C)_k = B_k + C_k$.

1.(brute-force):

$$\begin{aligned}\vec{A}\cdot(\vec{B}+\vec{C}) &= A_1(B_1+C_1)+A_2(B_2+C_2)+A_3(B_3+C_3)\\ &= A_1B_1+A_1C_1+A_2B_2+A_2C_2+A_3B_3+A_3C_3\\ &\equiv \vec{A}\cdot\vec{C}+\vec{A}\cdot\vec{B}.\end{aligned}$$

2.(Einstein): $\vec{A}\cdot(\vec{B}+\vec{C}) = A_m(B_m+C_m) = A_mB_m + A_mC_m \equiv \vec{A}\cdot\vec{C}+\vec{A}\cdot\vec{B}$.

***Example 4: Cross Products:*** to begin let me remind you of the definition of $\vec{A}\times\vec{B}$,

$$\vec{A}\times\vec{B} \equiv < A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1 >$$

Now there is clearly a pattern here, if we look at the first component of $\vec{A}\times\vec{B}$ is involves the second and third components of $\vec{A},\ \text{and}\ \vec{B}$. The same goes for the other slots, the $k-th$ component of the cross product involves the $i,j$-th components of $\vec{A},\ \text{and}\ \vec{B}$ where $i,j\neq k\ \text{and}\ i\neq j$.

***Definition:*** the antisymmetric (or Levi-Civita) symbol is $\epsilon_{ijk}$ where it is nonzero for $i,j,k$ distinct. In particular $\epsilon_{123}=\epsilon_{231}=\epsilon_{312}=1$ and $\epsilon_{321}=\epsilon_{213}=\epsilon_{132}=-1$ while,

$$\begin{aligned}\epsilon_{111}=\epsilon_{112}=\epsilon_{113}=\epsilon_{121}=\epsilon_{122}=\epsilon_{131}=\epsilon_{133}=0\\ \epsilon_{211}=\epsilon_{212}=\epsilon_{221}=\epsilon_{222}=\epsilon_{223}=\epsilon_{232}=\epsilon_{233}=0\\ \epsilon_{311}=\epsilon_{313}=\epsilon_{322}=\epsilon_{323}=\epsilon_{331}=\epsilon_{332}=\epsilon_{333}=0\end{aligned}$$

You can see that whenever an index is repeated in the antisymmetric symbol you get zero. More generally it is antisymmetric; $\epsilon_{ijk}=-\epsilon_{jik}$ and so forth for any exchange of two indices. The proof of antisymmetry follows from the numerical values listed above.

Now we can use the antisymmetric symbol to write the formula for the cross product neatly,

$$\begin{aligned}\vec{A}\times\vec{B} &= < \epsilon_{231}A_2B_3+\epsilon_{321}A_3B_2, \epsilon_{312}A_3B_1+\epsilon_{132}A_1B_3, \epsilon_{123}A_1B_2+\epsilon_{213}A_2B_1 >\\ &= (\epsilon_{231}A_2B_3+\epsilon_{321}A_3B_2)e_1 + (\epsilon_{312}A_3B_1+\epsilon_{132}A_1B_3)e_2 + (\epsilon_{123}A_1B_2+\epsilon_{213}A_2B_1)e_3\end{aligned}$$

Therefore,

$$\boxed{\vec{A}\times\vec{B} \equiv < A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1 > \iff \vec{A}\times\vec{B} = \epsilon_{ijk}A_iB_je_k}$$

In principle when we have 3 sums over indices which take 3 values there would be 3x3x3=27 terms but the antisymmetric symbol kills all the terms except the those terms for which the indices are distinct. We can also see from the expression above that

$$(\vec{A} \times \vec{B}) \cdot e_m = (\epsilon_{ijk} A_i B_j e_k) \cdot e_m = (\epsilon_{ijk} A_i B_j) e_k \cdot e_m = \epsilon_{ijm} A_i B_j$$

Because $e_i \cdot e_m = \delta_{im}$ which means that it is zero if $i \neq m$ and it is one if $i = m$. The index k was summed over to begin with but the dot product with $e_m$ picks out the $m - th$ component. Thus,

$$\boxed{(\vec{A} \times \vec{B})_k = \epsilon_{ijk} A_i B_j.}$$

***<u>Example 5: Skew Property of Cross Product:</u>*** Prove $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

1.)(brute-force)

$$\begin{aligned} \vec{A} \times \vec{B} &\equiv < A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1 > \\ &= - < A_3B_2 - A_2B_3, A_1B_3 - A_3B_1, A_2B_1 - A_1B_2 > \\ &= - < B_2A_3 - B_3A_2, B_3A_1 - B_1A_3, B_1A_2 - B_2A_1 > \\ &\equiv -\vec{B} \times \vec{A}. \end{aligned}$$

2.(Einstein) Observe $(\vec{A} \times \vec{B})_k = \epsilon_{ijk} A_i B_j = -\epsilon_{jik} A_i B_j = -\epsilon_{jik} B_j A_i = -(\vec{B} \times \vec{A})_k$. This holds for each $k = 1, 2, 3$ therefore $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

***<u>Example 6: Distributive Property of Cross Product:</u>*** Prove $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

1.)(brute-force) $\vec{A} \times (\vec{B} + \vec{C} \equiv$,

$$\begin{aligned} &\equiv < A_2(B_3 + C_3 - A_3(B_2 + C_2), A_3(B_1 + C_1) - A_1(B_3 + C_3), A_1(B_2 + C_2) - A_2(B_1 + C_1) > \\ &= < A_2B_3 + A_2C_3 - A_3B_2 - A_3C_2, A_3B_1 + A_3C_1 - A_1B_3 - A_1C_3, A_1B_2 + A_1C_2 - A_2B_1 - A_2C_1 > \\ &= < A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1 > + < A_2C_3 - A_3C_2, A_3C_1 - A_1C_3, A_1C_2 - A_2C_1 > \\ &\equiv \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \end{aligned}$$

2.)(Einstein) Note

$$(\vec{A} \times (\vec{B} + \vec{C}))_k = \epsilon_{ijk} A_i (B + C)_j = \epsilon_{ijk} A_i B_j + \epsilon_{ijk} A_i C_j = (\vec{A} \times \vec{B})_k + (\vec{A} \times \vec{C})_k$$

holds for each $k$ thus $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

***Example 7: Determinant of matrix:*** the antisymmetric symbol can be used to give compact formulas for the determinant of a matrix. The two-dimensional symbol is $\epsilon_{ij}$ which is nonzero only for $\epsilon_{12} = 1 \text{ and } \epsilon_{21} = -1$. Suppose we have a 2x2 matrix,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad det(A) = A_{11}A_{22} - A_{12}A_{21} = \frac{1}{2}A_{ij}A_{kl}\epsilon_{ik}\epsilon_{jl}$$

I'll show you the details of how that works out in office hours if you wish. There are similar formulas for larger matrices, this Levi-Civita symbol captures the algebraic structure of the determinant.

***Further Study:*** if you look through my posted homework solutions I discuss several identities for the Levi-Civita symbol. Almost every non-trivial vector identity I know of is a consequence of the combinatorics contained within the "Uber Lemma" (see H10). If you wish to see more surf around the ma430 page which is linked to my NCSU archive page. Probably Homework 1 would be good to tinker with if you want to get good at this stuff.

Calculations like these are at the heart of General Relativity which is wall to wall tensor analysis.

***Just so we are clear, this is not a required topic, you can do everything I ask via the brute-force method. However, I may opt to use Einstein notation in certain proofs.***

***For the tests, it is much more important that you master the homework and lecture examples.***