Parametrized Surfaces

We have seen that surfaces in $\mathbb{R}^3$ can be described as level surfaces ($x^2+y^2+z^2=1$ for example) or possibly as the graph of some function of two variables ($z = x+y$ for example). The case of the graph is really just a special case of the level surface idea, given $z = f(x,y)$ we can always construct $F(x,y,z) = z - f(x,y)$ and describe $z = f(x,y) \iff F(x,y,z) = 0$.

There is another approach to describing a surface in $\mathbb{R}^3$.

Def'n $\vec{X}: D \subseteq \mathbb{R}^2 \longrightarrow S \subseteq \mathbb{R}^3$ is a parametrization of the surface S. We say D is the parameter space and denote

$$\vec{X}(u,v) = (x(u,v), y(u,v), z(u,v))$$

the mapping $\vec{X}$ should be smooth

E78 The sphere $x^2+y^2+z^2=1$ has a natural parametrization in terms of the polar angle θ and azimuthal angle φ,

$$x = \cos\theta\sin\varphi \qquad y = \sin\theta\sin\varphi \qquad z = \cos\varphi$$

Notice that

$$\begin{aligned} x^2+y^2+z^2 &= \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi \\ &= \sin^2\varphi\,(\cos^2\theta + \sin^2\theta) + \cos^2\varphi \\ &= \sin^2\varphi + \cos^2\varphi \\ &= 1 \qquad \text{(this shows the parametrization works)} \end{aligned}$$

Thus $\vec{X}(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ parametrizes the sphere, we take $D = [0,2\pi]\times[0,\pi]$ in this case since we want our angles to range $0 \le \theta \le 2\pi$ and $0 \le \theta \le \pi$.

E79 Let $z = f(x,y)$ be a graph of some smooth function $f: D \to \mathbb{R}^2$ then we can choose x & y as our parameters and write

$$\vec{X}(x,y) = (x, y, f(x,y))$$

thus we find graphs are a special case of parametrically described surfaces.

E80 Consider $x^2+y^2-z^2=1$ this is a hyperboloid. If you know a little about hyperbolic trig. functions then the following parametrization is fairly obvious

$$\begin{aligned} x &= \cosh\gamma\cos\theta \\ y &= \cosh\gamma\sin\theta \\ z &= \sinh\gamma \end{aligned}$$

$$\begin{aligned} x^2+y^2-z^2 &= \cosh^2\gamma\cos^2\theta + \cosh^2\gamma\sin^2\theta - \sinh^2\gamma \\ &= \cosh^2\gamma - \sinh^2\gamma \\ &= 1. \quad \text{(it works)} \end{aligned}$$

We'd say γ is a hyperbolic angle, in contrast to the ordinary angle θ we must allow γ to range over $(-\infty,\infty)$ whereas $0\le\theta\le 2\pi$. The mapping in total is $\Sigma(\gamma,\theta)=(\cosh\gamma\cos\theta,\cosh\gamma\sin\theta,\sinh\gamma)$

Remark: you've probably noticed that we don't have to use "u" and "v" for the parameters. Likewise the mapping $\vec{\Sigma}$ is often replaced with $\vec{r}$ in the text or my homework sol^n's.

TANGENT PLANES:

We give the methodology for each description of a surface

I.) If $F(x,y,z)=0$ and (x_0,y_0,z_0) is a point on that surface then the tangent plane at (x_0,y_0,z_0) is

$$(\nabla F)(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0.$$

II.) If $z=f(x,y)$ and $z_0=f(x_0,y_0)$ then the tangent plane at (x_0,y_0,z_0),

$$z=z_0+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

this is really just a special case of I.) just take $F=f(x,y)-z$.

Tangent Planes Continued:

III.) Given a parametrized surface $\vec{\Sigma}(u,v)$ we study a particular point $\vec{\Sigma}(u_0, v_0)$. Notice we have two curves through the point which are constructed by fixing one of the parameters and letting the other vary; lets denote them $\vec{\alpha}$ & $\vec{\beta}$,

$$\vec{\alpha}(u) \equiv \vec{\Sigma}(u, v_0)$$
$$\vec{\beta}(v) \equiv \vec{\Sigma}(u_0, v)$$

The tangents vectors to these curves are given by the partial derivatives of the map $\vec{\Sigma}$,

$$\vec{\alpha}'(u) = \frac{\partial \vec{\Sigma}}{\partial u}(u, v_0) \quad \& \quad \vec{\beta}'(v) = \frac{\partial \vec{\Sigma}}{\partial v}(u_0, v)$$

then at the point of interest we have two tangents vectors to the surface and presumably $\vec{\alpha}'(u_0) \times \vec{\beta}'(v_0) \neq 0$, actually this is not universally true for all surfaces, but it is true for <u>oriented</u> surfaces where $\vec{\Sigma}_u \times \vec{\Sigma}_v \neq 0 \quad \forall u, v$.

$$\vec{N}(u_0, v_0) = \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \end{matrix} \right|_{(u_0, v_0)}$$

$$= \left\langle \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right\rangle \Bigg|_{(u_0, v_0)}$$

then we have a normal vector $\vec{N}(u_0, v_0)$ and a point so we construct the plane in the usual way

$$\vec{N}(u_0, v_0) \cdot \left(\vec{r} - \vec{\Sigma}(u_0, v_0) \right) = 0 \quad , \quad \vec{r} = \langle x, y, z \rangle .$$

Remark: its very simple, we get the normal one of two ways

(i) by a gradient ∇F normal to $F = 0$

(ii.) by cross product of tangents $\vec{N} = \vec{\Sigma}_u \times \vec{\Sigma}_v$ to $\vec{\Sigma}(u, v)$.